Standard Risk Aversion and Efficient Risk Sharing

Technical Appendix (Not for publication)

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The purpose of this technical appendix is to provide a detailed description of the numerical example presented in the paper. Specifically, we consider a couple who share risk efficiently between themselves and make joint investment decisions. The first agent is assumed to have constant-absolute-risk-aversion (CARA) utility,

$$u_1(c) = 1 - \frac{1}{\phi}\exp(-\phi c), \qquad \text{with } \phi > 0.$$

The second agent is assumed to have constant-relative-risk-aversion (CRRA) utility,

$$u_2(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \qquad \text{with } \sigma > 0.$$

The corresponding absolute risk tolerance and absolute prudence are given by

$$T_1(c) \equiv -\frac{u_1'(c)}{u_1''(c)} = \frac{1}{\phi}, \qquad T_2(c) \equiv -\frac{u_2'(c)}{u_2''(c)} = \frac{c}{\sigma},$$

$$P_1(c) \equiv -\frac{u_1'''(c)}{u_1''(c)} = \phi, \qquad P_2(c) \equiv -\frac{u_2'''(c)}{u_2''(c)} = \frac{1+\sigma}{c}.$$

Note that both agents have nondecreasing absolute risk tolerance and nonincreasing absolute prudence, hence both of them have standard risk aversion.

The representative agent's utility function $\widehat{u}(\cdot)$ is obtained by solving a resources allocation problem. Specifically, for any given Pareto weights, $\lambda_1 > 0$ and $\lambda_2 > 0$, and for any $z \geq 0$,

$$\widehat{u}(z) \equiv \max_{c_1, c_2} \{\lambda_1 u_1(c_1) + \lambda_2 u_2(c_2)\}$$

subject to $c_1 \geq 0$, $c_2 \geq 0$ and $c_1 + c_2 \leq z$. The solution of this problem involves a pair of sharing rules, $\kappa_1(z)$ and $\kappa_2(z)$, that are completely determined by

$$\widehat{u}'(z) = \lambda_1 \exp[-\phi \kappa_1(z)] = \lambda_2 [\kappa_2(z)]^{-\sigma},$$

$$\kappa_1(z) + \kappa_2(z) = z.$$

Using Equations (5) and (7) in the paper, we can derive the representative agent's absolute risk tolerance and absolute prudence, i.e.,

$$\widehat{T}(z) = T_1[\kappa_1(z)] + T_2[\kappa_2(z)] = \frac{1}{\phi} + \frac{1}{\sigma}\kappa_2(z) > 0,$$

$$\widehat{P}(z) = \frac{\{T_1[\kappa_1(z)]\}^2 P_1[\kappa_1(z)] + \{T_2[\kappa_2(z)]\}^2 P_2[\kappa_2(z)]}{\left[\widehat{T}(z)\right]^2} = \frac{\frac{1}{\phi} + \frac{1+\sigma}{\sigma^2}\kappa_2(z)}{\left[\frac{1}{\phi} + \frac{1}{\sigma}\kappa_2(z)\right]^2}.$$

Straightforward differentiation then yields

$$\widehat{P}'(z) = \frac{1}{\sigma}\frac{\kappa_2'(z)}{\left[\widehat{T}(z)\right]^3}\left[\frac{1}{\phi}\left(\frac{1}{\sigma} - 1\right) - \frac{1+\sigma}{\sigma^2}\kappa_2(z)\right].$$

Since $\sigma > 0$, $\kappa_2'(z) > 0$ and $\widehat{T}(z) > 0$,

$$\widehat{P}'(z; \lambda_1, \lambda_2) \gtrless 0 \qquad \text{if and only if} \qquad \frac{\sigma}{\phi}\left(\frac{1-\sigma}{1+\sigma}\right) \gtrless \kappa_2(z; \lambda_1, \lambda_2).$$

Thus, a necessary condition for $\widehat{P}'(z) > 0$ is $1 > \sigma$. Once this is granted, $\widehat{P}(z)$ is strictly increasing when $\kappa_2(z)$ is sufficiently small. The above condition also highlights the fact that the Pareto weights have a role in determining the slope of $\widehat{P}(z)$.

The main idea of the numerical examples is to show that when $\widehat{P}(z)$ is non-monotonic (or locally increasing), it is possible to find a pair of random variables $(\widetilde{x}, \widetilde{y})$ such that the couple (as a group) will invest more in the risky asset in the presence of background risk. In the following examples, we take as our benchmark case: $\lambda_1 = 1.5$, $\lambda_2 = 1.0$, $\phi = 0.1$ and $\sigma = 0.4$. Figure A1 plots the function $\widehat{P}(z)$ under three different values of σ, $\{0.3, 0.4, 0.5\}$. The other parameter values are as in the benchmark case. Figure A2 plots the function $\widehat{P}(z)$ under three different values of ϕ, and Figure A3 shows what happen to $\widehat{P}(z)$ when we change the value of λ_1. These diagrams show that the shape, as well as the level, of $\widehat{P}(z)$ is rather sensitive to changes in $\{\sigma, \phi, \lambda_1\}$. In all the case that we considered, $\widehat{P}(z)$ is increasing when z [and hence $\kappa_2(z)$] is small.

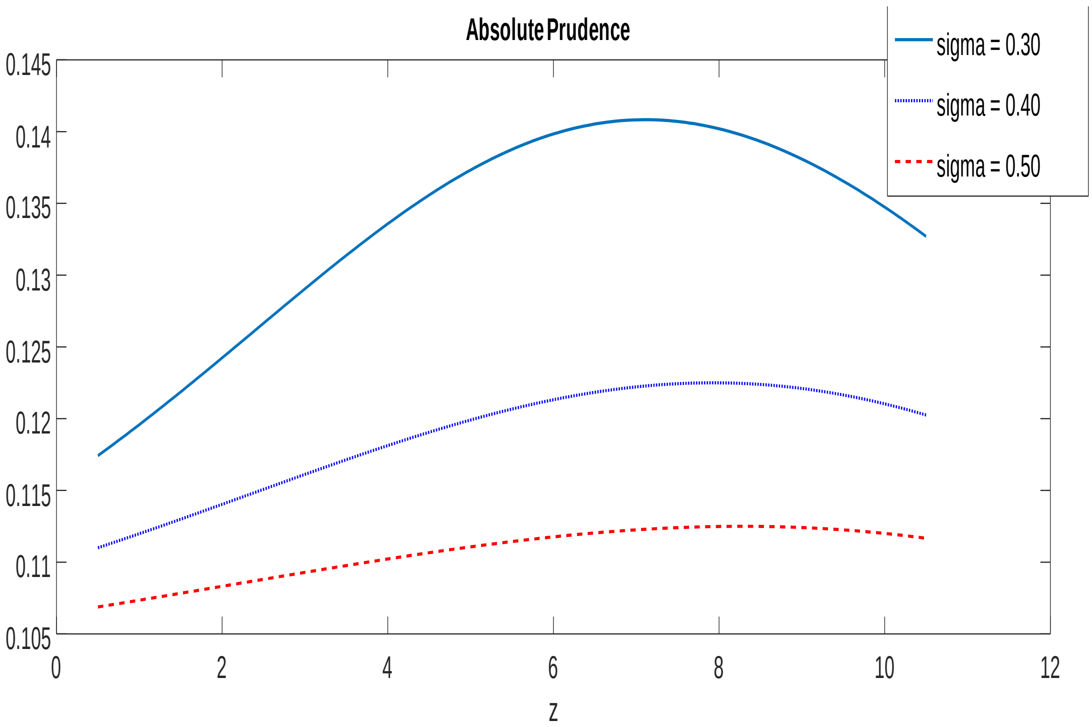

Figure A1

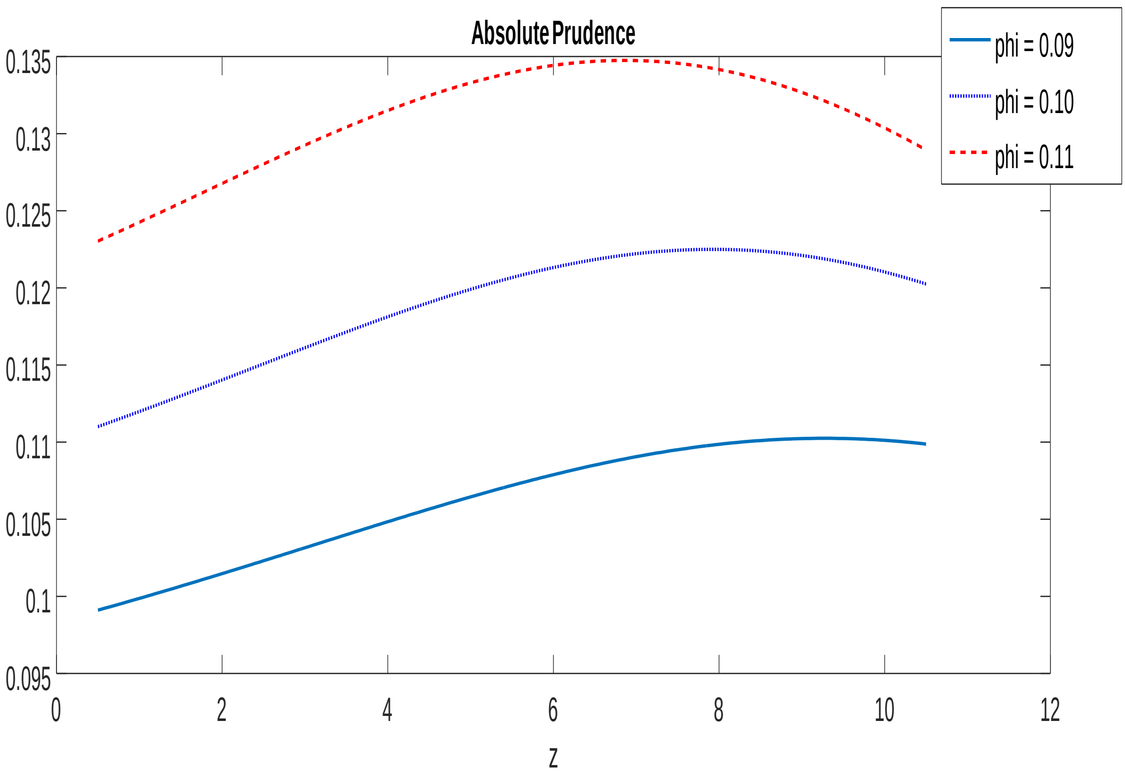

Figure A2

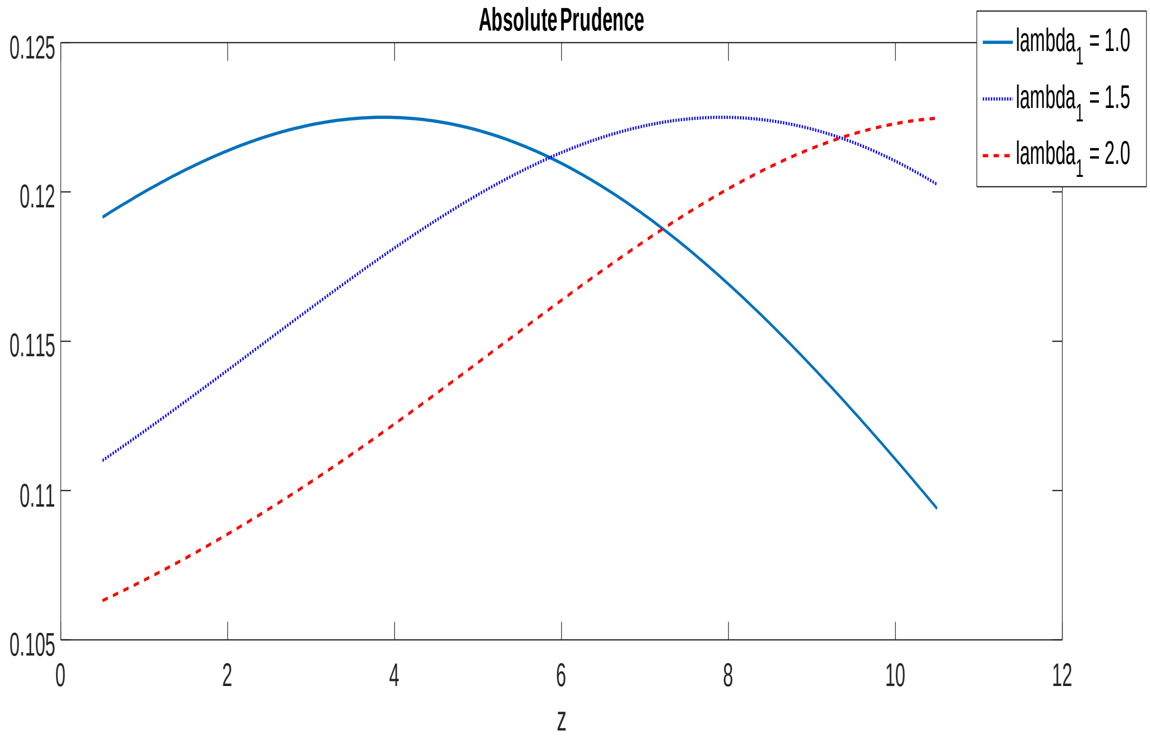

Figure A3

Suppose $\omega = 4.5$ and the excess returns of the risky asset has only two possible states, $\widetilde{x}_1 = -0.2$ and $\widetilde{x}_2 = 0.24$, with equal probability. Let α_1^* be the level of risky investment in the absence of background risk. This can be obtained by solving

$$\frac{1}{2}\widetilde{x}_1 \widehat{u}'\left(\omega + \alpha_1^* \widetilde{x}_1\right) + \frac{1}{2}\widetilde{x}_2 \widehat{u}'\left(\omega + \alpha_1^* \widetilde{x}_2\right) = 0$$

$$\Rightarrow \frac{\lambda_1}{2}\left\{\widetilde{x}_1 \exp\left[-\phi\kappa_1\left(\omega + \alpha_1^* \widetilde{x}_1\right)\right] + \widetilde{x}_2 \exp\left[-\phi\kappa_1\left(\omega + \alpha_1^* \widetilde{x}_2\right)\right]\right\} = 0.$$

The value of α_1^* is reported in Table A1.

Next, we introduce a background risk $\widetilde{y}$ which has three possible states: $\{-2, 0, 2.6\}$, with equal probability. In order to apply the variant of Proposition 6 in Kimball (1993, p.610), the following condition has to be satisfied

$$\frac{1}{6}\sum_{j=1}^{3}\sum_{i=1}^{2}\widehat{u}'\left(\omega + \alpha_1^* \widetilde{x}_i + \widetilde{y}_j\right) \geq \frac{1}{2}\sum_{i=1}^{2}\widehat{u}'\left(\omega + \alpha_1^* \widetilde{x}_i\right). \tag{C1}$$

This is verified in Table A1. Since $\widehat{u}(\cdot)$ has a decreasing absolute risk aversion, condition C1 is satisfied by any unfair background risk, i.e., $E(\widetilde{y}) \leq 0$. But this is *not a necessary condition* for

Table A1

	Benchmark	Changing σ		Changing ϕ		Changing λ_1	
σ	0.4	**0.3**	**0.5**	0.4	0.4	0.4	0.4
ϕ	0.1	0.1	0.1	**0.09**	**0.11**	0.1	0.1
λ_1	1.5	1.5	1.5	1.5	1.5	**1.0**	**2.0**
λ_2	1.0	1.0	1.0	1.0	1.0	1.0	1.0
ω	4.5	4.5	4.5	4.5	4.5	4.5	4.5
$E\left[\widehat{u}'\left(\omega+\alpha_1^*\widetilde{x}+\widetilde{y}\right)\right]$	0.6961	0.6966	0.6960	0.7160	0.6779	0.7719	0.6670
$E\left[\widehat{u}'\left(\omega+\alpha_1^*\widetilde{x}\right)\right]$	0.6957	0.6948	0.6965	0.7168	0.6762	0.7713	0.6671
Is C1 satisfied?	Yes	Yes	No	No	Yes	Yes	No
α_1^*	5.0928	5.3828			4.7860	6.1647	
α_2^*	5.1054	5.4015			4.7921	6.1545	

C1 to hold as attested by our results in Table A1. We are only interested in those cases in which condition C1 is satisfied. The optimal level of risky investment in the presence of background risk (denoted by α_2^*) is then obtained by solving

$$\frac{1}{6}\widetilde{x}_1\sum_{j=1}^{3}\widehat{u}'\left(\omega+\alpha_2^*\widetilde{x}_1+\widetilde{y}_j\right)+\frac{1}{6}\widetilde{x}_2\sum_{j=1}^{3}\widehat{u}'\left(\omega+\alpha_2^*\widetilde{x}_2\right)=0$$

$$\Rightarrow\frac{\lambda_1}{6}\left\{\widetilde{x}_1\sum_{j=1}^{3}\exp\left[-\phi\kappa_1\left(\omega+\alpha_2^*\widetilde{x}_1+\widetilde{y}_j\right)\right]+\widetilde{x}_2\sum_{j=1}^{3}\exp\left[-\phi\kappa_1\left(\omega+\alpha_2^*\widetilde{x}_2+\widetilde{y}_j\right)\right]\right\}=0. \quad \text{(Eq.1)}$$

Table A1 shows that in the benchmark scenario, i.e., $\sigma=0.4$, $\phi=0.1$ and $\lambda_1=1.5$, condition C1 is satisfied and the couple will increase their risky investment when there is background risk. Similar results can be obtained under three other combinations of $\{\sigma,\phi,\lambda_1\}$. Finally, Table A2 compares the couple's joint investment decision to those made the agents when they are acting alone. Note that the portfolio choice of agent 1 (with CARA utility) is unaffected by the background risk when acting alone. This can be easily seen by setting $\kappa_1(z)\equiv z$ in Eq.1. But agent 2, with strictly increasing absolute risk tolerance and strictly decreasing absolute prudence, will significantly lower his/her risky investment in the presence of background risk.

Table A2

	Joint Decision	Agent 1 alone	Agent 2 alone
σ	0.4	—	0.4
ϕ	0.1	0.1	—
ω	4.5	4.5	4.5
$E\left[\widehat{u}'\left(\omega+\alpha_1^*\widetilde{x}+\widetilde{y}\right)\right]$	0.6961	0.2823	0.6654
$E\left[\widehat{u}'\left(\omega+\alpha_1^*\widetilde{x}\right)\right]$	0.6957	0.2582	0.6430
Is C1 satisfied?	Yes	Yes	Yes
α_1^*	5.0928	1.3812	6.1971
α_2^*	5.1054	1.3812	5.0582