Mathematical Appendix

"Sources of Economic Growth in Models with Non-Renewable Resources"

A. Nested CES Production Functions

In this section, we will verify that Assumption A2 is satisfied by all the nested CES production functions considered in Sections 3 and 4 of the paper. We begin with the specification considered in Section 3, which is

$$F(K_t, Z_t) = \left[\alpha K_t^{\eta} + (1-\alpha) Z_t^{\eta}\right]^{\frac{1}{\eta}}, \qquad \text{with } \alpha \in (0,1) \text{ and } \eta < 1,$$

$$G(Q_t X_t, A_t N_t) \equiv \left[\varphi (Q_t X_t)^{\psi} + (1-\varphi)(A_t N_t) \quad \right]^{\frac{1}{}}, \qquad \text{with } \varphi \in (0,1) \text{ and } \psi < 1.$$

First, consider capital input. If $\eta \leq 0$, then

$$\lim_{K_t \to 0} F(K_t, G(Q_t X_t, A_t N_t)) = 0,$$

for any $Q_t X_t > 0$ and $A_t N_t > 0$, regardless of the value of ψ. Thus, physical capital is essential for production when $\eta \leq 0$. If $\eta \in (0,1)$, then

$$\lim_{K_t \to 0} F_1(K_t, G(Q_t X_t, A_t N_t)) = \infty,$$

regardless of the value of ψ. Next, consider the inputs of $G(\cdot)$. When $\psi \leq 0$, we have

$$\lim_{X_t \to 0} G(Q_t X_t, A_t N_t) = \lim_{N_t \to 0} G(Q_t X_t, A_t N_t) = 0,$$

$$\lim_{X_t \to 0} G_1(Q_t X_t, A_t N_t) = \varphi^{\frac{1}{}} \quad \text{and} \quad \lim_{N_t \to 0} G_2(Q_t X_t, A_t N_t) = (1-\varphi)^{\frac{1}{}}.$$

There are now two subcases to consider: If $\psi \leq 0$ and $\eta \leq 0$, then both natural resources and labour are essential for production, i.e.,

$$\lim_{X_t \to 0} F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right) = \lim_{N_t \to 0} F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right) = 0.$$

If $\psi \leq 0$ and $\eta \in (0,1)$, then we can show that

$$\begin{aligned} \lim_{X_t \to 0} \frac{\partial Y_t}{\partial X_t} &= (1-\alpha)\left\{\alpha \lim_{X_t \to 0}\left[\frac{G\left(Q_t X_t, A_t N_t\right)}{K_t}\right]^{-\eta} + 1 - \alpha\right\}^{\frac{1}{\eta}-1} \cdot \lim_{X_t \to 0} G_1\left(Q_t X_t, A_t N_t\right) \cdot Q_t \\ &= (1-\alpha) \cdot \infty \cdot \varphi^{\frac{1}{\psi}} Q_t = \infty. \end{aligned}$$

Likewise,

$$\begin{aligned} \lim_{N_t \to 0} \frac{\partial Y_t}{\partial N_t} &= (1-\alpha)\left\{\alpha \lim_{N_t \to 0}\left[\frac{G\left(Q_t X_t, A_t N_t\right)}{K_t}\right]^{-\eta} + 1 - \alpha\right\}^{\frac{1}{\eta}-1} \cdot \lim_{N_t \to 0} G_2\left(Q_t X_t, A_t N_t\right) \cdot A_t \\ &= (1-\alpha) \cdot \infty \cdot (1-\varphi)^{\frac{1}{\psi}} A_t = \infty. \end{aligned}$$

If $\psi \in (0,1)$, then we have

$$\lim_{X_t \to 0} G\left(Q_t X_t, A_t N_t\right) = (1-\varphi)^{\frac{1}{\psi}}\left(A_t N_t\right) \qquad \text{and} \qquad \lim_{N_t \to 0} G\left(Q_t X_t, A_t N_t\right) = \varphi^{\frac{1}{\psi}}\left(Q_t X_t\right),$$

$$\lim_{X_t \to 0} G_1\left(Q_t X_t, A_t N_t\right) = \lim_{N_t \to 0} G_2\left(Q_t X_t, A_t N_t\right) = \infty.$$

Using these we can obtain

$$\lim_{X_t \to 0} \frac{\partial Y_t}{\partial X_t} = F_2\left(K_t, (1-\varphi)^{\frac{1}{\psi}} A_t N_t\right)\left[\lim_{X_t \to 0} G_1\left(Q_t X_t, A_t N_t\right)\right] = \infty,$$

$$\lim_{N_t \to 0} \frac{\partial Y_t}{\partial N_t} = F_2\left(K_t, \varphi^{\frac{1}{\psi}} Q_t X_t\right)\left[\lim_{N_t \to 0} G_2\left(Q_t X_t, A_t N_t\right)\right] = \infty.$$

Note that these results hold regardless of the value of η.

Next, we turn to the production function in (44). There are now only two possible cases: If $\psi \leq 0$, then all three inputs are essential for production. If $\psi \in (0,1)$, then we can obtain

$$\lim_{N_t \to 0} \frac{\partial Y_t}{\partial N_t} = \varphi A_t \left\{\varphi + (1-\varphi) \lim_{N_t \to 0}\left[\frac{A_t N_t}{K_t^{\alpha}\left(Q_t X_t\right)^{1-\alpha}}\right]^{-\psi}\right\}^{\frac{1}{\psi}-1} = \infty,$$

$$\lim_{K_t \to 0} \frac{\partial Y_t}{\partial K_t} = \alpha (1-\varphi) \left\{ \varphi \lim_{N_t \to 0} \left[\frac{K_t^{\alpha} (Q_t X_t)^{1-\alpha}}{A_t N_t} \right]^{-\psi} + 1 - \varphi \right\}^{\frac{1}{\psi}-1} \left[\lim_{K_t \to 0} \left(\frac{K_t}{Q_t X_t} \right)^{\alpha-1} \right] = \infty,$$

$$\begin{aligned} \lim_{X_t \to 0} \frac{\partial Y_t}{\partial X_t} &= (1-\alpha)(1-\varphi) \left\{ \varphi \lim_{X_t \to 0} \left[\frac{K_t^{\alpha} (Q_t X_t)^{1-\alpha}}{A_t N_t} \right]^{-\psi} + 1 - \varphi \right\}^{\frac{1}{\psi}-1} \left[\lim_{X_t \to 0} \left(\frac{K_t}{Q_t X_t} \right)^{\alpha} \right] Q_t \\ &= \infty. \end{aligned}$$

Note that the production functions in (44) and (45) are essentially identical, except that $A_t N_t$ and $Q_t X_t$ have switched place. Thus, using the same line of argument we can show that (45) satisfies Assumption A2.

We now consider the production function in (46). The first thing to note is that labour input is essential for production regardless of the value of ψ. If $\psi \le 0$, then both physical capital and natural resources are essential for production. What remains is to consider the marginal product of these inputs when $\psi \in (0,1)$. Straightforward differentiation gives

$$\frac{\partial Y_t}{\partial K_t} = (1-\beta)\varphi \left(\frac{A_t N_t}{Q_t X_t} \right)^{\beta} \left[\varphi + (1-\varphi) \left(\frac{K_t}{Q_t X_t} \right)^{-\psi} \right]^{\frac{1}{\psi}-1} \left[\varphi \left(\frac{K_t}{Q_t X_t} \right)^{\psi} + 1 - \varphi \right]^{-\frac{\beta}{\psi}},$$

$$\frac{\partial Y_t}{\partial X_t} = (1-\beta)(1-\varphi) \left(\frac{A_t N_t}{K_t} \right)^{\beta} \left[\varphi \left(\frac{Q_t X_t}{K_t} \right)^{-\psi} + (1-\varphi) \right]^{\frac{1}{\psi}-1} \left[\varphi + (1-\varphi) \left(\frac{Q_t X_t}{K_t} \right)^{\psi} \right]^{-\frac{\beta}{\psi}}.$$

Using these and the following properties,

$$\lim_{K_t \to 0} \left[\varphi + (1-\varphi) \left(\frac{K_t}{Q_t X_t} \right)^{-\psi} \right]^{\frac{1}{\psi}-1} = \lim_{X_t \to 0} \left[\varphi \left(\frac{Q_t X_t}{K_t} \right)^{-\psi} + (1-\varphi) \right]^{\frac{1}{\psi}-1} = \infty,$$

$$\lim_{K_t \to 0} \left[\varphi \left(\frac{K_t}{Q_t X_t} \right)^{\psi} + 1 - \varphi \right]^{-\frac{\beta}{\psi}} = (1-\varphi)^{-\frac{\beta}{\psi}},$$

$$\lim_{X_t \to 0} \left[\varphi + (1-\varphi) \left(\frac{Q_t X_t}{K_t} \right)^{\psi} \right]^{-\frac{\beta}{\psi}} = \varphi^{-\frac{\beta}{\psi}},$$

we can get

$$\lim_{K_t \to 0} \frac{\partial Y_t}{\partial K_t} = \lim_{X_t \to 0} \frac{\partial Y_t}{\partial X_t} = \infty.$$

Since (46) and (47) are symmetric, the same line of argument can be used to show the desired properties for (47).

B. Proof of Theorem 3

We will consider each of the specifications in (44)-(47) separately. For each specification we will first verify the existence of a positive constant κ^* such that $K_t = \kappa^* Y_t$ for all t under conditions (vi)-(viii) in Section 3 of the paper.

Specification 1 We begin with the production function in (44). Under this specification, the first-order conditions for the representative firm's problem are given by

$$(1-\varphi)\,\alpha Y_t^{1-\psi} K_t^{\alpha\psi-1} (Q_t X_t)^{(1-\alpha)\psi} = r_t + \delta, \tag{B.1}$$

$$(1-\varphi)(1-\alpha)\, Y_t^{1-\psi} K_t^{\alpha\psi} (Q_t X_t)^{(1-\alpha)\psi-1} Q_t = (1+\mu)\, p_t, \tag{B.2}$$

$$\varphi Y_t^{1-\psi} (A_t N_t)^{\psi-1} A_t = w_t. \tag{B.3}$$

Combining (B.1) and (B.2) gives

$$\frac{p_t X_t}{K_t} = \frac{(1-\alpha)(r_t+\delta)}{\alpha(1+\mu)}. \tag{B.4}$$

Suppose conditions (vii) and (viii) are satisfied, i.e., $r_t = r^* > -\delta$ and $\tau_t = \tau^*$ for all t. Then both p_t and X_t are growing at some constant rate. It follows from (B.4) that K_t must also be growing at a constant rate. Next, dividing both sides of (15) by K_t gives

$$\frac{K_{t+1}}{K_t} = \frac{1}{2+\theta}\frac{w_t N_t}{K_t} - \frac{1-\tau_t}{\tau_t}\frac{p_t X_t}{K_t}. \tag{B.5}$$

If conditions (vii) and (viii) are satisfied, then τ_t, $p_t X_t / K_t$ and K_{t+1}/K_t are all constant over time. Hence, $w_t N_t / K_t$ must be constant over time as well. Finally, rewrite the production function in (44) as

$$Y_t^{\psi} = \varphi (A_t N_t)^{\psi} + (1-\varphi)\left[K_t^{\alpha} (Q_t X_t)^{1-\alpha}\right]^{\psi}.$$

Substituting (B.2) and (B.3) into this expression gives

$$Y_t^{\psi} = w_t N_t Y_t^{\psi-1} + \frac{1+\mu}{1-\alpha} p_t X_t Y_t^{\psi-1} \Longrightarrow \frac{Y_t}{K_t} = \frac{w_t N_t}{K_t} + \frac{1+\mu}{1-\alpha}\frac{p_t X_t}{K_t}.$$

This shows that Y_t/K_t is constant over time under conditions (vii) and (viii).

Substituting $r_t = r^*$ and $K_t = \kappa^* Y_t$ into (B.1) gives

$$(1-\varphi)\,\alpha\,(\kappa^*)^{\psi-1}\left(\frac{K_t}{Q_t X_t}\right)^{(\alpha-1)\psi} = (1-\varphi)\,\alpha\,(\kappa^*)^{\psi-1}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi} = r^* + \delta.$$

This shows that the ratio between $\widehat{k}_t$ and $\widehat{x}_t$ must be constant over time, or equivalently,

$$\frac{\widehat{x}_{t+1}}{\widehat{x}_t} = \frac{\widehat{k}_{t+1}}{\widehat{k}_t} = \frac{\gamma^*}{1+a} = \frac{(1+q)\,(1-\tau^*)}{(1+a)\,(1+n)}.$$

By the same token, we can also rewrite (B.2) and (B.3) as

$$(1+\mu)\,p_t = (1-\varphi)\,(1-\alpha)\,(\kappa^*)^{\psi-1}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi+1} Q_t, \tag{B.6}$$

$$w_t = \varphi\,(\kappa^*)^{\psi-1}\,\widehat{k}_t^{1-\psi} A_t. \tag{B.7}$$

Since the ratio between $\widehat{k}_t$ and $\widehat{x}_t$ is constant over time, it follows from (B.6) that p_t must be growing at the same rate as Q_t. By (4), we can write

$$\frac{p_{t+1}}{p_t} = 1 + r^* = \frac{Q_{t+1}}{Q_t} = 1 + q.$$

The last step is to substitute (B.6) and (B.7) into (16). This will give

$$(1+a)\,(1+n)\,\widehat{k}_{t+1} = (\kappa^*)^{\psi-1}\left[\frac{\varphi}{2+\theta}\widehat{k}_t^{1-\psi} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{(1-\varphi)\,(1-\alpha)}{1+\mu}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi}\widehat{k}_t\right]$$

$$\Rightarrow (1+a)\,(1+n)\,\frac{\widehat{k}_{t+1}}{\widehat{k}_t} = (\kappa^*)^{\psi-1}\left[\frac{\varphi}{2+\theta}\widehat{k}_t^{-\psi} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{(1-\varphi)\,(1-\alpha)}{1+\mu}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi}\right].$$

Since both $\widehat{k}_{t+1}/\widehat{k}_t$ and $\widehat{k}_t/\widehat{x}_t$ are constant over time, it follows that the *level* of $\widehat{k}_t$ must be constant over time in any equilibrium that satisfies conditions (vi)-(viii). Hence, we have $\gamma^* = 1+a$, $r^* = q$, and $(1-\tau^*) = (1+a)\,(1+n)\,/\,(1+q)$.

Specification 2 Consider the production function in (45). The first-order conditions for the firm's problem are now given by

$$(1-\varphi)\,\alpha Y_t^{1-\psi} K_t^{\alpha\psi-1}\,(A_t N_t)^{(1-\alpha)\psi} = r_t + \delta, \tag{B.8}$$

$$\varphi Y_t^{1-\psi}\left(Q_t X_t\right)^{\psi-1} Q_t = (1+\mu)\, p_t, \tag{B.9}$$

$$(1-\varphi)(1-\alpha)\, Y_t^{1-\psi} K_t^{\alpha\psi}\left(A_t N_t\right)^{\psi(1-\alpha)-1} A_t = w_t. \tag{B.10}$$

Combining (B.8) and (B.10) gives

$$\frac{w_t N_t}{K_t} = \frac{1-\alpha}{\alpha}\left(r_t+\delta\right),$$

which is constant over time under condition (vii). By assumption, both A_t and N_t grow at some exogenous constant rate. Condition (vi) implies that Y_t is growing at a constant rate, while condition (vii) states that r_t is time-invariant. Thus, it follows immediately from (B.8) that K_t must be growing at a constant rate. Equation (B.5) then implies that $p_t X_t / K_t$.must also be constant over time under conditions (vi)-(viii). Finally, rewrite the production function in (45) as

$$Y_t^{\psi} = \varphi\left(Q_t X_t\right)^{\psi} + (1-\varphi)\left[K_t^{\alpha}\left(A_t N_t\right)^{1-\alpha}\right]^{\psi}.$$

Substituting (B.8) and (B.9) into the above expression and rearranging terms gives

$$\frac{Y_t}{K_t} = (1+\mu)\frac{p_t X_t}{K_t} + \left(\frac{1-\varphi}{\alpha}\right)\left(r_t+\delta\right).$$

Thus, a constant r_t and a constant ratio $p_t X_t / K_t$ will imply a constant capital-output ratio.

Using the two conditions: $K_t = \kappa^* Y_t$ and $r_t = r^*$, we can rewrite the first-order conditions (B.8)-(B.10) as

$$\begin{aligned}
(1-\varphi)\left(\kappa^*\right)^{\psi-1}\alpha \widehat{k}_t^{(\alpha-1)\psi} &= r^*+\delta, \\
\varphi\left(\kappa^*\right)^{\psi-1}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{1-\psi} Q_t &= (1+\mu)\, p_t, \\
(1-\varphi)(1-\alpha)\left(\kappa^*\right)^{\psi-1}\widehat{k}_t^{(\alpha-1)\psi+1} A_t &= w_t.
\end{aligned} \tag{B.11}$$

The first one of these equations immediately implies that $\widehat{k}_t$ is constant over time, so that $\gamma^* = 1+a$. Substituting the last two equations into (16) gives

$$K_{t+1} = A_t N_t\left(\kappa^*\right)^{\psi-1}\left[\frac{(1-\varphi)(1-\alpha)}{2+\theta}\widehat{k}_t^{(\alpha-1)\psi+1} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{\varphi}{1+\mu}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{1-\psi}\widehat{x}_t\right]$$

$$\Rightarrow (1+a)(1+n)\,\widehat{k}_{t+1} = \left(\kappa^*\right)^{\psi-1}\left[\frac{(1-\varphi)(1-\alpha)}{2+\theta}\widehat{k}_t^{(\alpha-1)\psi+1} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{\varphi}{1+\mu}\widehat{k}_t^{1-\psi}\widehat{x}_t^{\psi}\right].$$

Since $\widehat{k}_t$ is constant over time, the above equation implies that $\widehat{x}_t$ must be constant over time as well. Finally, (B.11) implies that p_t must be growing at the same rate as Q_t in any equilibrium that satisfies conditions (vi)-(viii), so that $r^* = q$.

Specification 3 Next, we consider the production function in (46). The equilibrium factor prices are now characterised by

$$(1-\beta)\,\varphi\left[\varphi K_t^{\psi} + (1-\varphi)\,(Q_t X_t)^{\psi}\right]^{\frac{1-\beta}{\psi}-1} (A_t N_t)^{\beta}\, K_t^{\psi-1} = r_t + \delta, \tag{B.12}$$

$$(1-\beta)\,(1-\varphi)\left[\varphi K_t^{\psi} + (1-\varphi)\,(Q_t X_t)^{\psi}\right]^{\frac{1-\beta}{\psi}-1} (A_t N_t)^{\beta}\,(Q_t X_t)^{\psi-1}\, Q_t = (1+\mu)\, p_t, \tag{B.13}$$

$$\left[\varphi K_t^{\psi} + (1-\varphi)\,(Q_t X_t)^{\psi}\right]^{\frac{1-\beta}{\psi}} \beta\,(A_t N_t)^{\beta-1} A_t = w_t. \tag{B.14}$$

Combining (B.12) and (B.13) gives

$$\frac{p_t X_t}{K_t} = \frac{1-\varphi}{\varphi}\frac{(r_t+\delta)}{(1+\mu)}\left(\frac{Q_t X_t}{K_t}\right)^{\psi}. \tag{B.15}$$

Suppose both r_t and τ_t are constant over time. Then the above expression implies that K_t must be growing at a constant rate over time. From (B.14), we can get $\beta Y_t = w_t N_t$. Substituting this into (B.5) gives

$$\frac{K_{t+1}}{K_t} = \frac{\beta}{2+\theta}\frac{Y_t}{K_t} - \frac{1-\tau_t}{\tau_t}\frac{p_t X_t}{K_t}. \tag{B.16}$$

Finally, rewrite (B.12) to become

$$(1-\beta)\left(\frac{Y_t}{K_t}\right)\left[\frac{\varphi K_t^{\psi}}{\varphi K_t^{\psi} + (1-\varphi)\,(Q_t X_t)^{\psi}}\right] = (r_t+\delta)$$

$$\Longrightarrow \frac{Y_t}{K_t} = \frac{(r_t+\delta)}{1-\beta}\left[1 + \frac{1-\varphi}{\varphi}\left(\frac{Q_t X_t}{K_t}\right)^{\psi}\right] = \frac{r_t+\delta}{1-\beta} + \frac{1+\mu}{1-\beta}\frac{p_t X_t}{K_t}. \tag{B.17}$$

The second equality is obtained by using (B.15). Equations (B.16) and (B.17) now form a system of linear equations that can be used to solve for the value of Y_t/K_t and $p_t X_t/K_t$ in terms of K_{t+1}/K_t, τ_t and r_t. Since K_{t+1}/K_t, τ_t and r_t are all time-invariant under conditions (vi)-(viii), it follows that Y_t/K_t and $p_t X_t/K_t$ are also time-invariant.

Note that the condition $Y_t = \frac{1}{\kappa^*} K_t$ can be rewritten as

$$\left[\varphi \widehat{k}_t^{\psi} + (1-\varphi)\,\widehat{x}_t^{\psi}\right]^{\frac{1-\beta}{\psi}} = \frac{1}{\kappa^*}\widehat{k}_t$$

Using this, we can rewrite (B.12)-(B.14) as

$$(1-\beta)\,\varphi\,(\kappa^*)^{\frac{\psi}{1-\beta}-1}\,\widehat{k}_t^{-\frac{\beta\psi}{1-\beta}} = r_t + \delta,$$

$$(1-\beta)\,(1-\varphi)\,(\kappa^*)^{\frac{\psi}{1-\beta}-1}\,\widehat{k}_t^{1-\frac{\psi}{1-\beta}}\,\widehat{x}_t^{\psi-1} Q_t = (1+\mu)\,p_t,$$

$$\frac{1}{\kappa^*}\beta A_t \widehat{k}_t = w_t.$$

The first of these three equations, together with $r_t = r^*$, implies that $\widehat{k}_t$ must be constant over time. Hence, $\gamma^* = 1+a$. Substituting the last two equations into (16) gives

$$K_{t+1} = A_t N_t \left[\frac{\beta\widehat{k}_t}{(2+\theta)\,\kappa^*} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{(1-\beta)\,(1-\varphi)}{1+\mu}\,(\kappa^*)^{\frac{\psi}{1-\beta}-1}\,\widehat{k}_t^{1-\frac{\psi}{1-\beta}}\,\widehat{x}_t^{\psi}\right]$$

$$\Rightarrow (1+a)\,(1+n)\,\widehat{k}_{t+1} = \frac{\beta\widehat{k}_t}{(2+\theta)\,\kappa^*} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{(1-\beta)\,(1-\varphi)}{1+\mu}\,(\kappa^*)^{\frac{\psi}{1-\beta}-1}\,\widehat{k}_t^{1-\frac{\psi}{1-\beta}}\,\widehat{x}_t^{\psi}.$$

Since $\widehat{k}_t$ is constant over time, the above equation implies that $\widehat{x}_t$ must be constant over time as well. The remaining results follow by the same line argument as in Specification 2.

Specification 4 Finally, we consider the production function in (47). The first-order conditions for the firm's problem are now given by

$$(1-v)\,\varphi\,(Q_t X_t)^{v}\left[\varphi K_t^{\psi} + (1-\varphi)\,(A_t N_t)^{\psi}\right]^{\frac{1-v}{\psi}-1} K_t^{\psi-1} = r_t + \delta, \tag{B.18}$$

$$\nu\,(Q_t X_t)^{v-1}\,Q_t\left[\varphi K_t^{\psi} + (1-\varphi)\,(A_t N_t)^{\psi}\right]^{\frac{1-v}{\psi}} = (1+\mu)\,p_t, \tag{B.19}$$

$$(1-v)\,(1-\varphi)\,(Q_t X_t)^{v}\left[\varphi K_t^{\psi} + (1-\varphi)\,(A_t N_t)^{\psi}\right]^{\frac{1-v}{\psi}-1}(A_t N_t)^{\psi-1} A_t = w_t. \tag{B.20}$$

To start, using (47) and (B.19) we can obtain

$$\frac{p_t X_t}{K_t} = \frac{\nu}{1+\mu}\frac{Y_t}{K_t}.$$

Next, combining (B.18) and (B.20) gives

$$\frac{w_t N_t}{K_t} = \frac{(1-\varphi)\,(r_t+\delta)}{\varphi}\left(\frac{A_t N_t}{K_t}\right)^{\psi}. \tag{B.21}$$

Substituting these into (B.5) gives

$$\frac{K_{t+1}}{K_t} = \frac{(1-\varphi)(r_t+\delta)}{\varphi(2+\theta)} \left(\frac{A_t N_t}{K_t}\right)^{\psi} - \frac{\nu}{1+\mu} \left(\frac{1-\tau_t}{\tau_t}\right) \frac{Y_t}{K_t}. \tag{B.22}$$

We then use (B.18) to derive

$$\frac{Y_t}{K_t} = \left(\frac{r_t+\delta}{1-v}\right) \left[1 + \frac{1-\varphi}{\varphi} \left(\frac{A_t N_t}{K_t}\right)^{\psi}\right]. \tag{B.23}$$

Equations (B.22) and (B.23) form a system of linear equations which can be used to solve for Y_t/K_t and $(A_t N_t/K_t)^{\psi}$ in terms of K_{t+1}/K_t, r_t and τ_t. By conditions (vii) and (viii), both r_t and τ_t are time-invariant. Thus, what remains is to show that K_{t+1}/K_t is a constant under conditions (vi)-(viii). To this end, rewrite (B.19) as

$$\varphi K_t^{\psi} + (1-\varphi)(A_t N_t)^{\psi} = \left[\left(\frac{1+\mu}{\nu}\right) \frac{p_t}{Q_t}\right]^{\frac{\psi}{1-\nu}} (Q_t X_t)^{\psi}$$

and substitutes the above expression into (B.18) to get

$$(1-v)\varphi \left[\left(\frac{1+\mu}{\nu}\right) \frac{p_t}{Q_t}\right]^{\frac{1-\nu-\psi}{1-\nu}} (Q_t X_t)^{1-\psi} K_t^{\psi-1} = r_t + \delta.$$

The desired result follows from the fact that r_t is time-invariant and $\{p_t, X_t\}$ are growing at a constant rate under conditions (vii) and (viii). This proves that Y_t/K_t is a constant under conditions (vi)-(viii).

Next, we rewrite equations (B.18) and (B.19) as

$$(1-v)\varphi \widehat{x}_t^{v} \left(\varphi \widehat{k}_t^{\psi} + 1 - \varphi\right)^{\frac{1-v-\psi}{\psi}} \widehat{k}_t^{\psi-1} = r_t + \delta \tag{B.24}$$

$$v\frac{Y_t}{X_t} = v\widehat{x}_t^{v-1} \left(\varphi \widehat{k}_t^{\psi} + 1 - \varphi\right)^{\frac{1-v}{\psi}} Q_t = (1+\mu) p_t. \tag{B.25}$$

The condition $Y_t = \frac{1}{\kappa^*} K_t$ can be rewritten as

$$\widehat{x}_t^{v} \left(\varphi \widehat{k}_t^{\psi} + 1 - \varphi\right)^{\frac{1-v}{\psi}} = \frac{1}{\kappa^*} \widehat{k}_t. \tag{B.26}$$

Combining (B.24), (B.26) and $r_t = r^*$ gives

$$\frac{1}{\kappa^*} \frac{(1-v)\,\varphi \widehat{k}_t^{\psi}}{\varphi \widehat{k}_t^{\psi} + 1 - \varphi} = r^* + \delta$$

$$\Rightarrow (1-v)\,\varphi \widehat{k}_t^{\psi} = (r^* + \delta)\,\kappa^* \left(\varphi \widehat{k}_t^{\psi} + 1 - \varphi\right).$$

This can be used to derive a unique solution for $\widehat{k}_t$ which depends only on r^* and some parameters. Hence, $\gamma^* = 1 + a$. Equation (B.26) then implies that $\widehat{x}_t$ is also constant over time. Hence, $1 - \tau^* = (1+a)(1+n) / (1+q)$. Finally, given $\widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$, equation (B.25) implies that p_t and Q_t must be growing at the same rate. Hence, $r^* = q$.

This concludes the proof of Theorem 3.

C. Infinitely-Lived Consumers

In this section, we will show that the "knife-edge" condition of a unitary elasticity of substitution between effective labour input and effective resource input plays the same critical role in generating endogenous economic growth in an environment with infinitely-lived consumers. Specifically, an endogenous growth solution similar to the one in Agnani, Gutiérrez and Iza (2005) can be obtained when the elasticity of substitution of $G(\cdot)$ is identical to one. But if this elasticity is bounded away from one, then the common growth factor γ^* and interest rate r^* are solely determined by the growth rates of the exogenous technological factors (i.e., A_t and Q_t).

Consider an economy that is populated by $H > 0$ identical households. Each household contains a growing number of identical, infinitely-lived consumers. The size of each household at time t is given by $N_t = (1+n)^t$, with $n > 0$. Since all households are identical, we can focus on the choices made by a representative household and normalise H (which is just a scaling factor) to one. The representative household solves the following problem:

$$\max_{\{c_t, K_{t+1}, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to the sequential budget constraint

$$N_t c_t + K_{t+1} + p_t M_{t+1} = w_t N_t + (1 + r_t) K_t + p_t M_t,$$

where $\beta \in (0,1)$ is the subjective discount factor; $\sigma > 0$ is the reciprocal of the elasticity of intertemporal substitution (EIS); c_t denotes individual consumption at time t; K_t and M_t are, respectively, the household's holding of physical capital and non-renewable resources; p_t, w_t and r_t are as defined in Section 2.1 of the paper. The first-order conditions of this problem imply the Euler equation for consumption,

$$\frac{c_{t+1}}{c_t} = [\beta (1 + r_{t+1})]^{\frac{1}{\sigma}}, \tag{C.1}$$

and the Hotelling rule,

$$\frac{p_{t+1}}{p_t} = 1 + r_{t+1}.$$

We do not consider the resource tax in this setting (i.e., $\mu = 0$). The rest of the economy is the same as in Sections 2.2 and 2.3 of the paper. In particular, the first-order conditions for the firm's problem, (10)-(12), and the dynamic equation for natural resources, (13), remain unchanged. In any competitive equilibrium, goods market clear in every period so that

$$N_t c_t + K_{t+1} - (1-\delta) K_t = F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right), \qquad \text{for all } t \geq 0. \tag{C.2}$$

This replaces the capital market clearing condition in (16).

When characterising a balanced growth equilibrium, we maintain the three conditions (vi)-(viii) listed in Section 3. Note that Lemma 1 is also valid in this environment. First, consider the case when $G(\cdot)$ takes the Cobb-Douglas form, or equivalently, $\sigma_G(\cdot)$ is identical to one. Dividing both sides of (C.2) gives

$$\frac{N_t c_t}{K_t} + \frac{K_{t+1}}{K_t} - (1-\delta) = \frac{F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right)}{K_t}.$$

Hence, in any balanced growth equilibrium, aggregate consumption $N_t c_t$ must be growing at the same rate as K_t and Y_t. This, together with the Euler equation in (C.1) implies

$$\gamma^* = \left[\beta\left(1+r^*\right)\right]^{\frac{1}{\sigma}},$$

where γ^* is again the growth factor of per-capita output in a balanced growth equilibrium. Next, note that the arguments in Step 1 and Step 2 of the proof of Theorem 1 are built upon the properties of the production function and the characterising properties of balanced growth equilibrium. In particular, these arguments do not rely on the consumer side of the economy. Hence, they remain valid in this environment. Consequently, we have

$$\gamma^* = (1+b)\left(\frac{1-\tau^*}{1+n}\right)^{1-\phi},$$

$$\left(1+r^*\right)\left(1-\tau^*\right) = \gamma^*\left(1+n\right),$$

where $1+b \equiv (1+a)^{\phi}(1+q)^{1-\phi}$. Using these three equations, we can derive

$$1+r^* = \beta^{-\frac{\phi}{\varpi}}(1+b)^{\frac{\sigma}{\varpi}},$$

$$1-\tau^* = \beta^{\frac{1}{\varpi}}(1+b)^{\frac{1-\sigma}{\varpi}}(1+n),$$

$$\gamma^* = \beta^{\frac{1-\phi}{\varpi}}(1+b)^{\frac{\sigma}{\varpi}},$$

where $\varpi \equiv 1 - (1 - \sigma)(1 - \phi)$. Thus, a unique balanced growth equilibrium exists if

$$\beta^{\frac{1}{\varpi}} (1 + b)^{\frac{1-\sigma}{\varpi}} (1 + n) \in (0, 1),$$

which ensures that $\tau^* \in (0, 1)$. Notice that both γ^* and τ^* are endogenously determined by a host of factors as in the AGI solution.

Suppose now $\sigma_G(\cdot)$ is never equal to one. Since the arguments in Step 1 and Step 2 of the proof of Theorem 2 remain valid in this environment, we have $\gamma^* = 1 + a$, $r^* = q$, $\widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$. These in turn imply that

$$1 - \tau^* = \frac{(1 + a)(1 + n)}{1 + q}.$$