Bayesian Updating with Confounded Gaussian Signals

Let $\{X_j\}_{j=1}^{\infty}$ and $\{Y_j\}_{j=1}^{\infty}$ be two independent sequences of iid normal random variables drawn from the distributions $N\left(\theta_X^*, \sigma_X^2\right)$ and $N\left(\theta_Y^*, \sigma_Y^2\right)$, respectively. The variances σ_X^2 and σ_Y^2 are assumed to be known, but the means θ_X^* and θ_Y^* are not. The problem is to infer the unknown parameters from a sequence of confounded signals $\{R_j\}_{j=1}^{\infty}$, where $R_j = X_j + Y_j$ for all j. Since $X_j \perp Y_j$, each R_j is a normal random variable with mean $(\theta_X^* + \theta_Y^*)$ and variance $\sigma^2 \equiv \sigma_X^2 + \sigma_Y^2$.

Let $f\left(R_1, ..., R_j \mid \theta_X^*, \theta_Y^*\right)$ be the joint density function of $(R_1, ..., R_j)$ conditional on (θ_X^*, θ_Y^*). This satisfies

$$\begin{aligned}
f\left(R_1, ..., R_j \mid \theta_X^*, \theta_Y^*\right) &\propto \prod_{i=1}^{j} \exp\left\{-\frac{1}{2}\frac{\left[R_i - \left(\theta_X^* + \theta_Y^*\right)\right]^2}{\sigma^2}\right\} \\
&= \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^{j}\left[R_i - \left(\theta_X^* + \theta_Y^*\right)\right]^2}{\sigma^2}\right\} \\
&= \exp\left\{-\frac{1}{2}\left[\frac{\sum_{i=1}^{j} R_i^2 - 2j\overline{R}_j\left(\theta_X^* + \theta_Y^*\right) + j\left(\theta_X^* + \theta_Y^*\right)^2}{\sigma^2}\right]\right\}, \quad \text{(A.1)}
\end{aligned}$$

where $\overline{R}_j$ is the average value of $(R_1, ..., R_j)$. The prior distribution of (θ_X^*, θ_Y^*) is a bivariate normal distribution with mean vector and covariance matrix

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{X_0} \\ \mu_{Y_0} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \tau_{X_0}^2 & \rho_0 \tau_{X_0}\tau_{Y_0} \\ \rho_0 \tau_{X_0}\tau_{Y_0} & \tau_{Y_0}^2 \end{bmatrix}.$$

Let $g\left(\theta_X, \theta_Y\right)$ be the density function of the prior distribution, which satisfies

$$g\left(\theta_X, \theta_Y\right) \propto \exp\left\{-\frac{1}{2}\left[\frac{\left(\theta_X - \mu_{X_0}\right)^2}{\left(1-\rho_0^2\right)\tau_{X_0}^2} + \frac{\left(\theta_Y - \mu_{Y_0}\right)^2}{\left(1-\rho_0^2\right)\tau_{Y_0}^2} - \frac{2\rho_0\left(\theta_X - \mu_{X_0}\right)\left(\theta_Y - \mu_{Y_0}\right)}{\left(1-\rho_0^2\right)\tau_{X_0}\tau_{Y_0}}\right]\right\}. \quad \text{(A.2)}$$

After observing the realisation of $(R_1, ..., R_j)$, the density function of the posterior distribution is given by

$$h\left(\theta_X, \theta_Y \mid R_1, ..., R_j\right) = f\left(R_1, ..., R_j \mid \theta_X, \theta_Y\right) g\left(\theta_X, \theta_Y\right).$$

The task here is to express this density function in the following form

$$h\left(\theta_X, \theta_Y \mid R_1, ..., R_j\right) \propto \exp\left\{-\frac{1}{2}\left[a\theta_X^2 - 2b\theta_X + c\theta_Y^2 - 2d\theta_Y - 2e\theta_X\theta_Y\right]\right\}.$$

Once the coefficients (a, b, c, d, e) are determined, the updated moments can be obtained by using the formulae in Lemma 4.1 in Farzinnia and McCardle (2010, p.967).

It suffice to consider the sum of the terms in the square brackets in (A.1) and (A.2), i.e.,

$$\begin{aligned}
& \frac{\sum_{i=1}^{j} R_i^2 - 2j\overline{R}_j \left(\theta_X + \theta_Y\right) + j\left(\theta_X + \theta_Y\right)^2}{\sigma^2} \\
& + \frac{\left(\theta_X - \mu_{X_0}\right)^2}{\left(1-\rho_0^2\right)\tau_{X_0}^2} + \frac{\left(\theta_Y - \mu_{Y_0}\right)^2}{\left(1-\rho_0^2\right)\tau_{Y_0}^2} - \frac{2\rho_0\left(\theta_X - \mu_{X_0}\right)\left(\theta_Y - \mu_{Y_0}\right)}{\left(1-\rho_0^2\right)\tau_{X_0}\tau_{Y_0}} \\
= & \frac{\sum_{i=1}^{j} R_i^2 - 2j\overline{R}_j \left(\theta_X + \theta_Y\right) + j\left(\theta_X^2 + 2\theta_X\theta_Y + \theta_Y^2\right)}{\sigma^2} \\
& + \frac{\left(\theta_X^2 - 2\mu_{X_0}\theta_X + \mu_{X_0}^2\right)}{\left(1-\rho_0^2\right)\tau_{X_0}^2} + \frac{\left(\theta_Y^2 - 2\mu_{Y_0}\theta_Y + \mu_{Y_0}^2\right)}{\left(1-\rho_0^2\right)\tau_{Y_0}^2} - \frac{2\rho_0\left(\theta_X\theta_Y - \mu_{X_0}\theta_Y - \mu_{Y_0}\theta_X + \mu_{X_0}\mu_{Y_0}\right)}{\left(1-\rho_0^2\right)\tau_{X_0}\tau_{Y_0}} \\
= & \left[\frac{j}{\sigma^2} + \frac{1}{\left(1-\rho_0^2\right)\tau_{X_0}^2}\right]\theta_X^2 - 2\left[\frac{j\overline{R}_j}{\sigma^2} + \frac{\mu_{X_0}}{\left(1-\rho_0^2\right)\tau_{X_0}^2} - \frac{\rho_0\mu_{Y_0}}{\left(1-\rho_0^2\right)\tau_{X_0}\tau_{Y_0}}\right]\theta_X \\
& + \left[\frac{j}{\sigma^2} + \frac{1}{\left(1-\rho_0^2\right)\tau_{Y_0}^2}\right]\theta_Y^2 - 2\left[\frac{j\overline{R}_j}{\sigma^2} + \frac{\mu_{Y_0}}{\left(1-\rho_0^2\right)\tau_{Y_0}^2} - \frac{\rho_0\mu_{X_0}}{\left(1-\rho_0^2\right)\tau_{X_0}\tau_{Y_0}}\right]\theta_Y \\
& -2\left[-\frac{j}{\sigma^2} + \frac{\rho_0}{\left(1-\rho_0^2\right)\tau_{X_0}\tau_{Y_0}}\right]\theta_X\theta_Y + \kappa,
\end{aligned}$$

where κ is a constant term that does not depend on (θ_X, θ_Y). Hence, we can get

$$a = \frac{j}{\sigma^2} + \frac{1}{\left(1-\rho_0^2\right)\tau_{X_0}^2},$$

$$b = \frac{j\overline{R}_j}{\sigma^2} + \frac{\mu_{X_0}}{\left(1-\rho_0^2\right)\tau_{X_0}^2} - \frac{\rho_0\mu_{Y_0}}{\left(1-\rho_0^2\right)\tau_{X_0}\tau_{Y_0}},$$

$$c = \frac{j}{\sigma^2} + \frac{1}{\left(1-\rho_0^2\right)\tau_{Y_0}^2},$$

$$d = \frac{j\overline{R}_j}{\sigma^2} + \frac{\mu_{Y_0}}{\left(1-\rho_0^2\right)\tau_{Y_0}^2} - \frac{\rho_0\mu_{X_0}}{\left(1-\rho_0^2\right)\tau_{X_0}\tau_{Y_0}},$$

$$e = -\frac{j}{\sigma^2} + \frac{\rho_0}{\left(1-\rho_0^2\right)\tau_{X_0}\tau_{Y_0}}.$$

The correct formulae in Theorem 4.2 can be obtained by substituting these into the formulae in Lemma 4.1. The correct formulae in Theorem 4.1 can be obtained by setting $j = 1$.