Finite State Markov-chain Approximations to Highly Persistent Processes*

Karen A. Kopecky† Richard M. H. Suen‡

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Abstract

The Rouwenhorst method of approximating stationary AR(1) processes has been overlooked by much of the literature despite having many desirable properties unmatched by other methods. In particular, we prove that it can match the conditional and unconditional mean and variance, and the first-order autocorrelation of any stationary AR(1) process. These properties make the Rouwenhorst method more reliable than others in approximating highly persistent processes and generating accurate model solutions. To illustrate this, we compare the performances of the Rouwenhorst method and four others in solving the stochastic growth model and an income fluctuation problem. We find that (i) the choice of approximation method can have a large impact on the computed model solutions, and (ii) the Rouwenhorst method is more robust than others with respect to variation in the persistence of the process, the number of points used in the discrete approximation and the procedure used to generate model statistics.

Keywords: Numerical Methods, Finite State Approximations

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†Department of Economics, Social Science Center, Room 4701, The University of Western Ontario, London, Ontario, N6A 5C2, Canada. Email: *kkopecky@uwo.ca*.

‡Corresponding author. Department of Economics, Sproul Hall, University of California, Riverside CA 92521-0427. Email: *richard.suen@ucr.edu*. Tel.: (951) 827-1502. Fax: (951) 827-5685.

1 Introduction

In macroeconomic models, the exogenous stochastic process is typically assumed to follow a stationary first-order autoregressive process. For instance, in the standard real business cycle model the logarithm of the productivity shock is assumed to follow a Gaussian AR(1) process. When solving these models numerically, the continuous-valued autoregressive process is usually replaced by a discrete state-space Markov chain. To this end, researchers typically employ the approximation method proposed by Tauchen (1986), or the quadrature-based method developed in Tauchen and Hussey (1991). For AR(1) processes with low persistence, these methods can produce highly accurate approximations. However, their performance deteriorates when the serial correlation is very close to one.[1]

These findings raise concerns because macroeconomic studies typically employ highly persistent processes. In particular, there are two main questions that await answers. First, is there a more reliable technique to approximate highly persistent processes? Second, and more importantly, how does the performance of these methods affect the computed solutions of macroeconomic models? In quantitative studies, approximating the exogenous process is seldom an end in itself. Thus a more appropriate metric for evaluating approximation methods would be their impact on the computed solutions of the entire model. To the best of our knowledge, no existing studies have performed this kind of evaluation. The current study is intended to fill this gap.

The main objective of this paper is to answer the above questions. Regarding the first question, this paper re-examines a Markov-chain approximation method that is first proposed in Rouwenhorst (1995). The main strength of this method is that it can match five important statistics of *any* stationary AR(1) process, including the conditional and unconditional mean, the conditional and unconditional variance, and the first-order autocorrelation. This property makes the Rouwenhorst method more reliable than the other methods in approximating highly persistent processes. The Rouwenhorst method is particularly suited to approximate Gaussian AR(1) processes. This is because under this method the invariant distribution of the Markov chain converges to a normal distribution when the number of states is made sufficiently large. The first contribution of this

[1] This weakness is also acknowledged in the original papers. In Tauchen (1986, p.179), the author notes that "Experimentation showed that the quality of the approximation remains good except when λ [the serial correlation] is very close to unity." In Tauchen and Hussey (1991), the authors note that for processes with high persistence, "adequate approximation requires successively finer state spaces."

paper is to provide formal proofs of these results.[2]

Our second contribution is to compare the performances of five different approximation methods in solving two common macroeconomic models. The methods under study include the Tauchen (1986) method, the original Tauchen-Hussey method, a variation of this method proposed by Flodén (2008a), the Adda-Cooper (2003) method and the Rouwenhorst method. The first model that we consider is the prototypical stochastic neoclassical growth model without leisure. The neoclassical growth model is by far the most common analytical framework in macroeconomics. It is also often used as the test model for comparing solution methods.[3] Most importantly, it is possible to derive closed-form solutions for the neoclassical growth model under certain specifications. This property of the model provides tremendous convenience for evaluating the accuracy of the approximation methods. The main evaluation criterion in this application is the accuracy in approximating the business cycle moments generated by the model. The second model that we consider is an income fluctuation problem. This problem is of interest because it forms an integral part of the heterogeneous-agent models considered in Aiyagari (1994) and Krusell and Smith (1998). There is now a large literature that uses these models to examine issues in macroeconomics and finance. In terms of computation, the occasionally binding borrowing constraint in this problem makes it more challenging to solve than the stochastic growth model. In this application, the five methods are evaluated for their accuracy in approximating the degree of inequality in consumption, income and assets. In both models, we use two different approaches to compute the statistics of interest. In the baseline approach, an approximation for the stationary distribution of the state variables is first derived. The statistics are then computed directly from this distribution. In the second approach, the statistics are generated using Monte Carlo simulations. More specifically, we draw a common set of realizations from the actual AR(1) process and compute the statistics using the computed policy functions.

Our main findings from the stochastic growth model are as follows. Regardless of which approach is taken, the choice of approximation method can have a large impact on the accuracy of the computed business cycle moments. Under the baseline approach, a method that generates a good

[2]Some of these features are briefly mentioned in Rouwenhorst (1995). But a formal proof of these results is still lacking.

[3]The version that we consider is also used in Taylor and Uhlig (1990) and the companion papers to illustrate and compare different solution methods. More recently, Aruoba *et al.* (2006) use the stochastic growth model, but with labor-leisure choice, to compare different solution methods.

approximation for the AR(1) process also tends to yield accurate approximations for the business cycle moments. The Rouwenhorst method has the best performance in this regard. Furthermore, the high degree of accuracy of the Rouwenhorst method prevails even when a coarse state space (with only five states for the exogenous shock) is used. An improved version of the Tauchen (1986) method has the second best overall performance. In the sensitivity analysis, it is shown that the superior performance of the Rouwenhorst method is robust under a wide range of parameter values.

When the Monte Carlo simulation method is used to generate the business cycle moments, no single method dominates all others in all cases. With a logarithmic utility function and full depreciation, the five methods yield almost identical results. When a more realistic value of the depreciation rate is used, the Rouwenhorst method can again produce highly accurate approximations when there are only five states in the Markov chain. The other methods require a much finer state space (at least 25 states) in order to produce the same precision as the Rouwenhorst method. Another interesting finding is that the baseline approach, equipped with the Rouwenhorst method, performs as well as the simulation method in generating the business cycle moments. This result is of interest because the simulation method is considered standard practice in estimating unknown statistics of stochastic models. However, our results show that a high degree of accuracy in the business cycle moments generated from the neoclassical growth model can be achieved without simulation.

As for the income fluctuation problem, our results show that, consistent with the findings for the growth model, the methods which generate good approximations for the AR(1) process also tend to yield more accurate solutions under the baseline approach. When the persistence of the AR(1) process is set to 0.9, the Tauchen-Hussey method, Flodén's variation and the Rouwenhorst method have the best performance in this regard. When the persistence of the AR(1) process is increased to 0.977, Flodén's variation and the Rouwenhorst method continue to have the best performance but the accuracy of the Tauchen-Hussey method declines significantly. Moreover, the Rouwenhorst method is again found to be the most robust with respect to a reduction in the number of states in the Markov chain. In addition, the Rouwenhorst method is the only method that produces very similar, yet relatively accurate, results under both the baseline approach and the simulation approach. This shows that the Rouwenhorst method is also the least sensitive to changes in the procedure used to compute the statistics from the stationary distribution.

In sum, our quantitative results have two main implications. First, the accuracy of the approximation for the exogenous process can have a large impact on the computed solutions of macroeconomic models. Thus caution must be taken when choosing an approximation method. Second, our results show that the Rouwenhorst method is the most robust of the five methods considered with respect to the degree of persistence of the AR(1) process, the coarseness of the state-space of the discrete approximation, and the approach used to compute the statistics from the stationary distribution. The accuracies of model solutions computed using the Tauchen (1986) method and the Tauchen-Hussey method, on the other hand, are both sensitive to these choices. It is also worth noting that the performance of the Tauchen (1986) method is extremely sensitive to the choice of a free parameter that determines the bounds on the state-space of the discrete process. We show that choosing this parameter such that the unconditional variances of the discrete and continuous process coincide greatly increases the performance of this method. This feature of the Tauchen (1986) method is not documented by the existing literature.

The current study is related to Flodén (2008a) and Lkhagvasuren and Galindev (2008). The objective of Flodén (2008a) is to compare the relative performance of various discretization methods in approximating stationary AR(1) processes. However, Flodén does not consider the Rouwenhorst method, nor does he consider the impact of the discretization procedure on the solutions of macroeconomic models. The main objective of Lkhagvasuren and Galindev (2008) is to develop an approximation method for vector autoregressive processes with correlated error terms. Under the proposed method, the original multivariate process is decomposed into a number of independent univariate AR(1) processes. These independent processes are then approximated using the conventional methods. Lkhagvasuren and Galindev show, through a few numerical examples, that the Rouwenhorst method outperforms other methods in approximating moments of univariate AR(1) processes. In contrast, the current study formally proves that the Rouwenhorst method can be used to match the key statistics of any stationary AR(1) process.

The rest of this paper is organized as follows. Section 2 presents the Rouwenhorst method and its main features. Section 3 evaluates the performance of the Rouwenhorst method and four other discretization methods in solving the stochastic growth model and the income fluctuation problem. Section 4 concludes.

2 The Rouwenhorst Method

Consider the AR(1) process

$$z_t = \rho z_{t-1} + \varepsilon_t, \tag{1}$$

where $|\rho| < 1$ and ε_t is a white noise process with variance σ_ε^2. The AR(1) process is covariance-stationary with mean zero and variance $\sigma_z^2 = \sigma_\varepsilon^2 / \left(1 - \rho^2\right)$. If, in addition, ε_t is normally distributed in each period, then z_t is also normally distributed.

Rouwenhorst (1995) proposes a method to approximate this stochastic process by a discrete state-space process $\{y_t\}$. This involves constructing an N-state Markov chain characterized by (i) a symmetric and evenly-spaced state space $Y_N = \{\overline{y}_1, ..., \overline{y}_N\}$, with $\overline{y}_1 = -\psi$ and $\overline{y}_N = \psi$, and (ii) a transition matrix Θ_N. For any $N \geq 2$, the transition matrix Θ_N is determined by two parameters, $p, q \in (0, 1)$, and is defined recursively as follows:

Step 1: When $N = 2$, define Θ_2 as

$$\Theta_2 = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}.$$

Step 2: For $N \geq 3$, construct the N-by-N matrix

$$p \begin{bmatrix} \Theta_{N-1} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} + (1-p) \begin{bmatrix} \mathbf{0} & \Theta_{N-1} \\ 0 & \mathbf{0}' \end{bmatrix} + (1-q) \begin{bmatrix} \mathbf{0}' & 0 \\ \Theta_{N-1} & \mathbf{0} \end{bmatrix} + q \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \Theta_{N-1} \end{bmatrix},$$

where $\mathbf{0}$ is a $(N-1)$-by-1 column vector of zeros.

Step 3: Divide all but the top and bottom rows by two so that the elements in each row sum to one.

The main objective of this section is to show formally that the Rouwenhorst method has a number of desirable features that are unmatched by other methods. However, the matrix Θ_N generated by the three-step procedure above is difficult to work with analytically. Thus, we begin our analysis by offering a new, analytically tractable procedure for generating the Rouwenhorst

matrix. The main advantage of this new procedure is that it greatly simplifies the proofs of our analytical results.

2.1 Reconstructing the Rouwenhorst Matrix

For any $p, q \in (0,1)$ and for any integer $N \geq 2$, define a system of polynomials as follows

$$\Phi(t; N, i) \equiv [p + (1-p)\, t]^{N-i} (1 - q + qt)^{i-1}, \tag{2}$$

for $i = 1, 2, ..., N$. Expanding the polynomials in (2) yields

$$\Phi(t; N, i) = \sum_{j=1}^{N} \pi_{i,j}^{(N)} t^{j-1}, \qquad \text{for } i = 1, 2, ..., N. \tag{3}$$

Define an N-by-N matrix $\Pi_N = \left[\pi_{i,j}^{(N)}\right]$ using the coefficients in (3). Using the generating function in (2), one can derive the elements in Π_N recursively using the elements in Π_{N-1}, for $N - 1 \geq 2$. The details of this procedure are described in Appendix A. The main result of this subsection is Proposition 1 which states that the matrix Π_N is identical to the Rouwenhorst matrix Θ_N for any integer $N \geq 2$. All proofs can be found in Appendix B.

Proposition 1 *For any $N \geq 2$ and for any $p, q \in (0,1)$, the matrix Π_N defined above is identical to the Rouwenhorst matrix Θ_N generated by Steps 1-3.*

The next result states that Π_N is a stochastic matrix of non-zero entries. To begin with, set $t = 1$ in both (2) and (3) to obtain

$$\sum_{j=1}^{N} \pi_{i,j}^{(N)} = 1, \qquad \text{for } i = 1, 2, ..., N.$$

This means the elements in any row of Π_N sum to one. If, in addition, $\pi_{i,j}^{(N)} \geq 0$ for all i and j, then Π_N is a stochastic matrix. This is proved in the following lemma.

Lemma 2 *For any $N \geq 2$, the matrix Π_N defined above is a stochastic matrix of non-zero entries.*

Table 1: Selected Moments of the Markov Chain

Conditional Mean	$\mathrm{E}(y_{t+1}\|y_t = \overline{y}_i)$	$(q-p)\,\psi + (p+q-1)\,\overline{y}_i$
Conditional Variance	$\mathrm{var}(y_{t+1}\|y_t = \overline{y}_i)$	$\frac{4\psi^2}{(N-1)^2}\left[(N-i)\,(1-p)\,p + (i-1)\,q\,(1-q)\right]$
Unconditional Mean	$\mathrm{E}(y_t)$	$\frac{(q-p)\psi}{2-(p+q)}$
Unconditional Second Moment	$\mathrm{E}\left(y_t^2\right)$	$\psi^2\left\{1 - 4s\,(1-s) + \frac{4s(1-s)}{N-1}\right\}$
First-order Autocovariance	$\mathrm{Cov}(y_t, y_{t+1})$	$(p+q-1)\mathrm{var}(y_t)$
First-order Autocorrelation	$\mathrm{Corr}(y_t, y_{t+1})$	$p+q-1$

2.2 Discrete State-Space Markov Chain

Consider a Markov chain $\{y_t\}$ with a symmetric and evenly-spaced state space $Y_N = \{\overline{y}_1, ..., \overline{y}_N\}$ defined over the interval $[-\psi, \psi]$. The transition matrix of the Markov chain is given by $\Pi_N = \left[\pi_{i,j}^{(N)}\right]$ as defined above. The following result follows immediately from Lemma 2.

Proposition 3 *For any $N \geq 2$, the Markov chain with state space Y_N and transition matrix Π_N has a unique invariant distribution $\boldsymbol{\lambda}^{(N)} = \left(\lambda_1^{(N)}, ..., \lambda_N^{(N)}\right)$, where $\lambda_i^{(N)} \geq 0$ and $\sum_{i=1}^N \lambda_i^{(N)} = 1$.*

Rouwenhorst mentions that in the symmetric case where $p = q$, the unique invariant distribution is a binomial distribution with parameters $N-1$ and $1/2$. Our next result shows that the unique invariant distribution is binomial for any $p, q \in (0,1)$. Since the invariant distribution is unique, it can be solved by the guess-and-verify method. Let $s \equiv \frac{1-q}{2-(p+q)} \in (0,1)$. The guess for $\boldsymbol{\lambda}^{(N)}$, represented by $\widehat{\boldsymbol{\lambda}}^{(N)}$, is a binomial distribution with parameters $N-1$ and $1-s$. This means

$$\widehat{\lambda}_i^{(N)} = \binom{N-1}{i-1} s^{N-i}\,(1-s)^{i-1}, \qquad \text{for } i = 1, 2, ..., N. \tag{4}$$

It is easy to check that this is the actual solution when $N = 2$. The result for the general case is established in Proposition 4.

Proposition 4 *For any $N \geq 2$, the invariant distribution of the Markov chain defined above is a binomial distribution with parameters $N-1$ and $1-s$.*

Equipped with the invariant distribution, one can derive the unconditional moments of the Markov chain. Some of these moments are shown in Table 1. The mathematical derivations of these results can be found in Appendix C.

2.3 Approximating AR(1) Processes

The task at hand is to approximate a given stationary AR(1) process with an N-state Markov chain.[4] Let $\{z_t\}$ be the stationary AR(1) process defined in (1). Conditional on the realization of z_{t-1}, the mean and variance of z_t are given by ρz_{t-1} and σ_ε^2, respectively. Now define an N-state discrete Markov process $\{y_t\}$ as in Section 2.2 with

$$p = q = \frac{1+\rho}{2} \qquad \text{and} \qquad \psi = \sqrt{N-1}\sigma_\varepsilon. \tag{5}$$

Using the equations in Table 1, it is immediate to see that the resulting Markov chain has the same unconditional mean, unconditional variance and first-order autocorrelation as $\{z_t\}$. Suppose $y_{t-1} = \overline{y}_i$ for some $\overline{y}_i$ in Y_N. The conditional mean and conditional variance of y_t are given by

$$\mathrm{E}\left(y_t | y_{t-1} = \overline{y}_i\right) = \rho \overline{y}_i \qquad \text{and} \qquad \mathrm{var}\left(y_t | y_{t-1} = \overline{y}_i\right) = \sigma_\varepsilon^2.$$

Thus $\{y_t\}$ also has the same conditional mean and conditional variance as $\{z_t\}$.

Two remarks regarding this procedure are worth mentioning. First, under the Rouwenhorst method, the approximate Markov chain is constructed using ρ and σ_ε^2 alone. In particular, the transition matrix Π_N is not a discretized version of the conditional distribution of z_t. This is the fundamental difference between this method and the ones proposed in Tauchen (1986) and Tauchen and Hussey (1991). Second, the above procedure can be applied to *any* stationary AR(1) process, including those with very high persistence. Thus, unlike the other two methods, the one proposed by Rouwenhorst can always match the unconditional variance and the persistence of $\{z_t\}$.

Since the invariant distribution of $\{y_t\}$ is a binomial distribution with mean zero and variance $\sigma_y^2 = \sigma_\varepsilon^2/(1-\rho^2)$, the standardized process $\{y_t/\sigma_y\}$ converges to the standard normal distribution as N goes to infinity. Thus the Rouwenhorst method is particularly apt for approximating Gaussian AR(1) processes.

[4] In this paper, we focus on univariate AR(1) processes only. For vector autoregressive processes, one can combine the Rouwenhorst method with the decomposition method proposed in Lkhagvasuren and Galindev (2008). More specifically, these authors propose a method to decompose a multivariate process into a number of independent univariate processes. These independent processes can then be approximated using the Rouwenhorst method described below.

3 Evaluations

In this section we examine the performance of the Rouwenhorst method and four other discretization methods in solving the stochastic growth model and the income fluctuation problem. For the stochastic growth model, the main evaluation criterion is the accuracy in approximating the business cycle moments generated by the model. For the income fluctuation problem, we focus on measures of inequality in consumption, income and assets. The other methods under evaluation are described below.

Tauchen (1986) method Under this method, an evenly-spaced state space $Y_N = \{\overline{y}_1, ..., \overline{y}_N\}$ is used to construct the Markov chain $\{y_t\}$, with

$$\overline{y}_N = -\overline{y}_1 = \Omega\sigma_z,$$

where Ω is a positive real number and σ_z is the standard deviation of the original AR(1) process. Let Φ be the probability distribution function for the standard normal distribution. For any $i = 1, ..., N$, the transition probabilities of the Markov chain are given by

$$\pi_{i,j} = \Phi\left(\frac{\overline{y}_j - \rho\overline{y}_i + h/2}{\sigma_\varepsilon}\right),$$

for $j = 1$ and N, and

$$\pi_{i,j} = \Phi\left(\frac{\overline{y}_j - \rho\overline{y}_i + h/2}{\sigma_\varepsilon}\right) - \Phi\left(\frac{\overline{y}_j - \rho\overline{y}_i - h/2}{\sigma_\varepsilon}\right),$$

for $j = 2, ..., N-1$, where h is the step size between the grid points. It turns out that the performance of this method is strongly affected by the choice of Ω. To the best of our knowledge, there is no established rule for determining this parameter. Tauchen (1986) sets $\Omega = 3$ without giving any justification. Flodén (2008a) sets $\Omega = 1.2\ln(N)$. As explained in the results section, Flodén's choice of Ω is the main reason why he finds that the Tauchen (1986) method performs poorly in approximating highly persistent processes. In all the results reported below, Ω is calibrated such that the standard deviation of $\{y_t\}$ matches the standard deviation of the AR(1) process. This gives the method its best chance in approximating the AR(1) process. We choose to target σ_z instead of

ρ because, relative to σ_z, the persistence parameter ρ is well approximated under this method for a range of values of Ω and degrees of persistence.

The Quadrature-Based Methods Under this class of methods, the elements of the state space $Y_N = \{\overline{y}_1, ..., \overline{y}_N\}$ are determined by

$$\overline{y}_i = \sqrt{2}\sigma x_i, \qquad \text{for } i = 1, 2, ..., N,$$

where $\{x_i\}$ are the Gauss-Hermite nodes defined on $[-\infty, \infty]$. Let $\{\phi_j\}$ be the corresponding Gauss-Hermite weights. The elements in the transition matrix Π are then given by

$$\pi_{i,j} = \frac{f\left(\overline{y}_j|\overline{y}_i\right)}{f\left(\overline{y}_j|0\right)}\frac{\overline{w}_j}{s_i},$$

where $\overline{w}_j = \phi_j/\sqrt{\pi}$, the function $f\left(\cdot|\overline{y}_i\right)$ is the density function for $N\left(\rho\overline{y}_i, \sigma^2\right)$, and

$$s_i = \sum_{n=1}^{N} \frac{f\left(\overline{y}_n|\overline{y}_i\right)}{f\left(\overline{y}_n|0\right)}\overline{w}_n.$$

In Tauchen and Hussey (1991), the standard deviation σ is taken to be σ_ε. In Flodén (2008a), σ is a weighted average of σ_z and σ_ε. In particular, $\sigma = \omega\sigma_\varepsilon + (1-\omega)\sigma_z$ with $\omega = 0.5 + 0.25\rho$.

The Adda-Cooper (2003) Method The first step of this method is to partition the real line into N intervals. These intervals are constructed so that the random variable z_t has an equal probability of falling into them. Formally, let $I_n = [x_n, x_{n+1}]$ be the nth interval with $x_1 = -\infty$ and $x_{N+1} = +\infty$. The cut-off points $\{x_n\}_{n=2}^{N}$ are the solutions of the following system of equations:

$$\Phi\left(\frac{x_{n+1}}{\sigma_z}\right) - \Phi\left(\frac{x_n}{\sigma_z}\right) = \frac{1}{N}, \qquad \text{for } n = 1, 2, ..., N,$$

where Φ is the probability distribution function for the standard normal distribution. The nth element in the state space $Y_N = \{\overline{y}_1, ..., \overline{y}_N\}$ is the mean value of the nth interval. For any $i, j \in \{1, 2, ..., N\}$, the transition probability $\pi_{i,j}$ is defined as the probability of moving from interval I_i to interval I_j in one period.

3.1 Stochastic Growth Model

Consider the planner's problem in the stochastic growth model,

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \log (C_t) \right]$$

subject to

$$C_t + K_{t+1} = \exp (a_t) K_t^{\alpha} + (1 - \delta) K_t,$$

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}, \qquad \text{with } \rho \in (0, 1), \tag{6}$$

$C_t, K_{t+1} \geq 0$, and K_0 given, where C_t denotes consumption at time t, K_t denotes capital, $A_t \equiv \exp (a_t)$ is the technological factor and $\varepsilon_{t+1} \sim$ i.i.d. $N\left(0, \sigma_{\varepsilon}^2\right)$. The parameter $\beta \in (0, 1)$ is the subjective discount factor, $\alpha \in (0, 1)$ is the share of capital income in total output and $\delta \in (0, 1]$ is the depreciation rate. For any given value of a, define $\overline{K}(a)$ by

$$\overline{K}(a) = \left[\frac{\exp (a)}{\delta} \right]^{\frac{1}{1-\alpha}}.$$

Then, conditional on $a_t = a$, the state space of capital can be restricted to $\mathcal{K}(a) = \left[0, \overline{K}(a)\right]$. The state space of the stochastic growth model is given by

$$\mathcal{S} = \{(K, a) : K \in \mathcal{K}(a), a \in \mathbb{R}\}.$$

Let $\Psi (K, a)$ be the feasible choice set for next-period capital when the current state is (K, a). Formally, this is defined as

$$\Psi (K, a) = \left\{K' : \exp (a) K^{\alpha} + (1 - \delta) K \geq K' \geq 0\right\}.$$

The Bellman equation for this problem is

$$V (K, a) = \max_{K' \in \Psi(K,a)} \left\{ \log \left(\exp (a) K^{\alpha} + (1 - \delta) K - K' \right) + \beta \int V \left(K', a'\right) dF \left(a' | a\right) \right\}, \tag{7}$$

where $F(\cdot|a)$ is the distribution function of a_{t+1} conditional on $a_t = a$. The solution of this problem

includes a value function $V : \mathcal{S} \to \mathbb{R}$ and a policy function for next-period capital $g : \mathcal{S} \to \mathbb{R}$. With logarithmic utility function and full depreciation, the policy function and the stationary distribution can be derived analytically. Specifically, the policy function for next-period capital (in logarithmic terms) is given by

$$k_{t+1} = g(k_t, a_t) \equiv \ln \alpha\beta + a_t + \alpha k_t. \tag{8}$$

The stationary distribution of (k, a) is a bivariate normal distribution with mean vector

$$\boldsymbol{\mu}' = \begin{bmatrix} \frac{\ln(\alpha\beta)}{1-\alpha} & 0 \end{bmatrix},$$

and variance-covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_k^2 & \sigma_{ka} \\ \sigma_{ka} & \sigma_a^2 \end{bmatrix},$$

where

$$\sigma_k^2 = \frac{(1+\alpha\beta)\,\sigma_a^2}{(1-\alpha^2)\,(1-\alpha\rho)},$$

$$\sigma_{ka} = \frac{\rho\sigma_a^2}{1-\alpha\rho}, \qquad \text{and} \qquad \sigma_a^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2}.$$

Using these closed-form solutions, we can derive analytically the business cycle moments. These results are then used to assess the relative performance of five different discretization methods.

Parameterization and Computation

The first step in computing the business cycle moments is to assign values for the parameters $\{\alpha, \beta, \delta, \sigma_\varepsilon, \rho\}$. In the baseline scenario, we set $\delta = 1$ so that there is full depreciation. The full depreciation assumption is later relaxed. The other parameter values are chosen to be the same as in King and Rebelo (1999): $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$ and $\rho = 0.979$.

The next step is to discretize the state space $\mathcal{S}$. First, the AR(1) process in (6) is approximated using the methods mentioned above. The resulting N-state Markov chain is characterized by a state space $\mathcal{A} = \{\overline{a}_1, ..., \overline{a}_N\}$ and a transition matrix $\Pi = [\pi_{i,j}]$. Second, the continuous state space for capital is replaced by an evenly-spaced grid. Define the variable $k \equiv \ln K$. The set of grid points

for k is represented by $\mathcal{K} = \left\{\overline{k}_1, ..., \overline{k}_M\right\}$. The discretized state space can be expressed by

$$\widehat{\mathcal{S}} = \left\{\left(\overline{k}_m, \overline{a}_n\right) : \overline{k}_m \in \mathcal{K}, \overline{a}_n \in \mathcal{A}\right\}. \tag{9}$$

In the baseline case, the number of states in the Markov chain is set to five and the number of grid points for capital is 1000. After the discrete state space $\widehat{\mathcal{S}}$ is formed, the value function and the associated policy function are solved using the value-function iteration method described in Tauchen (1990) and Burnside (1999). The outcome is a discrete approximation to the policy function, denoted by $\left\{\widehat{g}\left(\overline{k}_m, \overline{a}_n\right) : \left(\overline{k}_m, \overline{a}_n\right) \in \widehat{\mathcal{S}}\right\}$.

The business cycle moments are then computed using two different approaches. Under the baseline approach, an approximation to the stationary distribution of the state variables (k, a) is first computed. To achieve this, we need to construct the transition matrix for these variables. Under the discrete state-space method, the probability of moving from state $\left(\overline{k}_m, \overline{a}_n\right)$ in $\widehat{\mathcal{S}}$ to state $\left(\overline{k}_l, \overline{a}_j\right)$ in $\widehat{\mathcal{S}}$ in one period is specified by

$$\Pr\left[\left(k', a'\right) = \left(\overline{k}_l, \overline{a}_j\right) | \left(k, a\right) = \left(\overline{k}_m, \overline{a}_n\right)\right] = \begin{cases} \pi_{n,j}, & \text{if } \overline{k}_l = \widehat{g}\left(\overline{k}_m, \overline{a}_n\right) \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

The resulting NM-by-NM transition matrix is denoted P. Let $\widehat{\boldsymbol{\pi}} = (\widehat{\pi}_1, ..., \widehat{\pi}_{NM})$ be the stationary distribution associated with P. Formally, this is defined by

$$\widehat{\boldsymbol{\pi}} P = \widehat{\boldsymbol{\pi}}.$$

In principle, $\widehat{\boldsymbol{\pi}}$ can be obtained as the eigenvector of P corresponding to eigenvalue 1, with the normalization $\sum_{i=1}^{NM} \widehat{\pi}_i = 1$. This method, however, is not practical when the number of grid points in the state space is large. In the following experiments, an approximation for the stationary distribution is obtained by iterating on the equation

$$\widetilde{\boldsymbol{\pi}}^l P = \widetilde{\boldsymbol{\pi}}^{l+1}. \tag{11}$$

The iterations proceed until the "distance" between successive iterates, as measured by $\max \left|\widetilde{\boldsymbol{\pi}}^l - \widetilde{\boldsymbol{\pi}}^{l+1}\right|$,

is within the desired tolerance. Given the approximate stationary distribution $\widetilde{\boldsymbol{\pi}}^l$ and the policy function $\widehat{g}$, the business cycle moments of interest can be computed.

Under the second approach, the business cycle moments are generated using Monte Carlo simulations. The standard procedure involves the following steps. Draw a common sequence of pseudo-random numbers of length $T = 5,010,000$ for the disturbance term ε.[5] Construct the random variable a_t using the *actual* AR(1) process given in (6). The resulting sequence is denoted $\{\widetilde{a}_t\}_{t=0}^{T}$. Construct a sequence of capital $\left\{\widetilde{k}_t\right\}_{t=0}^{T}$ according to

$$\widetilde{k}_{t+1} = \widehat{g}\left(\widetilde{k}_t, \widetilde{a}_t\right), \quad \text{with } \widetilde{k}_0 \text{ given.}$$

In general, the generated values of $\widetilde{k}_t$ and $\widetilde{a}_t$ will not coincide with the grid points in $\widehat{\mathcal{S}}$. In this case, linear interpolation is used to compute the value of $\widehat{g}\left(\widetilde{k}_t, \widetilde{a}_t\right)$. To ensure that the generated values of $\widetilde{k}_t$ and $\widetilde{a}_t$ are drawn from the stationary distribution, the first $10,000$ observations in either sequence are deleted. The remaining five million observations are used to compute the business cycle moments.

Baseline Results

Table 2 presents the baseline results. The five discretization methods are compared on three grounds: (i) the accuracy in approximating the AR(1) process, (ii) the precision in approximating the stationary distribution of the state variables, and (iii) the accuracy in approximating the business cycle moments. The table gives the ratio of the statistics computed following the above procedure to their *true* values. The true values are derived using the closed-form solutions mentioned above.

Panel (A) of Table 2 shows the performance of these methods in approximating the AR(1) process. As explained in Section 2.3, the transition matrix in the Rouwenhorst method (R) can be calibrated to match exactly the persistence parameter, the standard deviation of ε and the standard deviation of a. Similarly the parameter Ω in the Tauchen (1986) method is calibrated to match exactly the standard deviation of a_t. The required value is $\Omega = 1.6425$. With this choice of Ω, the Tauchen (1986) method yields a very small relative error (less than one percent) in approximating

[5] Specifically, we use the Mersenne Twister random number generator to generate the pseudorandom numbers. The generated sequence is first adjusted to remove any first-order serial correlation in it that may be introduced by the pseudorandom number generator. The resulting sequence is then transformed to one with mean zero and variance σ^2.

the persistence parameter. Our results are in stark contrast to those reported in Flodén (2008a) Table 2. Using $\Omega = 1.9313$ and $N = 5$, Flodén shows that this method produces a 12 percent error in approximating σ_a and a 1.5 percent error in approximating ρ. These results illustrate that the performance of the Tauchen (1986) method is very sensitive to the choice of Ω.

Next, we consider the accuracies of these methods in approximating the stationary distribution of the state variables. Panel (B) of Table 2 shows the performance of these methods in approximating the standard deviation of k and the covariance between a and k. In general, a discretization method that generates an accurate approximation for σ_a also has high precision in approximating these two moments. Among these five methods, the Rouwenhorst method has the highest accuracy in approximating these two moments. The relative errors for the two are about 0.14 percent. The Tauchen (1986) method is the second best. These two methods outperform the others by a significant margin.

Next, we compare the performance of these methods in approximating the business cycle moments. In particular, we focus on the standard deviation of output, consumption and investment (in logarithmic terms) and the first-order autocorrelation of output (in logarithmic terms).[6] The results are shown in panel (C) of Table 2. Again the Rouwenhorst method has the best overall performance in terms of approximating all these moments. However, with $\Omega = 1.6425$, the Tauchen (1986) method can produce highly accurate approximations that are comparable to those generated by the Rouwenhorst method. As mentioned above, the performance of this method is very sensitive to the choice of Ω. If we set $\Omega = 1.9313$ as in Flodén (2008a), then the Tauchen (1986) method would generate a 12-percent error in approximating the standard deviations.[7]

Finally, two things can be observed when comparing across all three panels. First, the relative errors in approximating σ_a are very similar to those in approximating the standard deviation of capital, output, consumption and investment. Second, the relative errors in approximating ρ are close to those in approximating the first-order autocorrelation for output. These results suggest that a good approximation for the moments of the AR(1) process is important in obtaining an accurate approximation for the business cycle moments.

[6] The first-order autocorrelation of consumption and investment (in logarithmic terms), and the cross-correlation between output and these variables are not shown in the paper. These results are available from the authors upon request.

[7] These results are not shown in here but are available from the authors upon request.

Error Analysis

The relative errors reported in Table 2 have a number of sources. For the purpose of this discussion, we classify these into two groups. The first group of errors arises when solving the Bellman equation in (7). This includes the errors that arise when we restrict the choice of next-period capital to a discrete set of values, and the truncation errors that emerge when we approximate the fixed point of the Bellman equation using a finite number of iterations. The second group of errors occurs during the computation of the stationary distribution of the state variables. First, the transition matrix P, constructed using the discrete Markov chain and the computed policy function, is an approximation of the actual transition function. Second, truncation errors arise when we approximate the stationary distribution using a finite number of iterations. The second group of errors would not occur if Monte Carlo simulations are used to generate the business cycle moments. The simulation approach, however, suffers from two other types of errors. First, approximation errors arise when we use linear interpolation to compute the values of $\widehat{g}(k,a)$ for points outside the discrete state space. Second, sampling errors arise when we compute the business cycle moments using finite samples.[8]

Using the actual policy function, it is possible to disentangle the two groups of errors in the baseline approach. Consider the following experiment. Construct a discrete state space $\widehat{\mathcal{S}}$ as in (9) using one of the five discretization methods. Construct the transition matrix P as in (10) *but replace the computed policy function* $\widehat{g}(k,a)$ *with the actual one in (8).* Iterate equation (11) successively to obtain an approximation for the stationary distribution of the state variables. Finally, use the approximate stationary distribution and the *actual* policy function $g(k,a)$ to compute the business cycle moments. By replacing $\widehat{g}(k,a)$ with the actual policy function, this procedure effectively removes all the errors involved in solving the Bellman equation. The remaining errors are thus due to the approximation of the stationary distribution of the state variables. The results of this procedure are reported in panel (B) of Table 3. To facilitate comparison, the baseline results are shown in panel (A) of the same table.

It is immediate to see that the figures in the two panels are almost identical. Replacing the

[8] By the Law of Large Numbers, the sampling error goes to zero as the number of observations approaches infinity. However, since our samples are finite, some sampling errors remain. These errors are small relative to the errors caused by linear interpolation.

computed policy function with the actual one does not affect the approximation of the technology shock process. As a result, the approximated values for ρ, σ_ε and σ_a are identical in the two sets of results. As for the standard deviations of the endogenous variables, only minor discrepancies are observed in the two panels. In other words, even though we have removed all the errors in computing the policy function, the baseline results remain largely unchanged. This has two implications. First, this implies that almost all the relative errors in the baseline case are due to the approximation of the stationary distribution $\widehat{\boldsymbol{\pi}}$. Second, this means the choice of discretization method has only a relatively minor impact on the solution of the Bellman equation. In sum, this experiment illustrates that the choice of discretization method matters because it would significantly affect the approximation of the stationary distribution.

The same conclusion can be drawn from another experiment. Suppose now the business cycle moments are computed using Monte Carlo simulations. More specifically, after solving the dynamic programming problem in (7), the model is simulated using the actual AR(1) process and the computed policy function $\widehat{g}(k, a)$. Under this procedure, the choice of discretization method only affects the simulated moments through the computed policy function. Table 4 presents the relative errors obtained under this procedure alongside with the baseline results. The two methods of generating business cycle moments have produced very different results. When the model is simulated using the actual AR(1) process, all five discretization methods generate almost identical results. This again implies that the differences in the baseline results across the five discretization methods are due to the approximation of the stationary distribution $\widehat{\boldsymbol{\pi}}$.

Finally, when comparing between the two panels of Table 4, one can see that the baseline approach, when combined with the Rouwenhorst method, can generate estimated moments that are as accurate as those produced by the simulation method with five million draws.

Robustness Check

In this section, it is shown that the relative performance of the five discretization methods are robust to changes in (i) the number of points in the discrete state space N, (ii) the persistence parameter ρ, and (iii) the standard deviation of the white noise process σ_ε.

Changing the Number of States Table 5 compares the performance of the five methods under different choices of N. Notice that the superior performance of the Rouwenhorst method is robust even when there are only two states in the discrete Markov chain. The relative errors in approximating the standard deviations of output, capital, consumption and investment are similar in all three cases. In particular, increasing the number of states from five to ten increases the precision only marginally. The original Tauchen-Hussey method has the lowest precision among the five in all three cases. Even when the number of states is increased to ten, the Tauchen-Hussey method can only replicate 57 percent of the actual value of σ_y. The performance of this method is much better when approximating ρ_y but the precision is still the lowest among the five.

Next, we consider the performance of the Tauchen (1986) method. For each value of N, we adjust the parameter Ω so as to match the actual value of σ_a. The required values for $N = 2$ and $N = 10$ are 1.0000 and 1.9847, respectively. In other words, in order to match the standard deviation σ_a, a wider state space (i.e., a larger value of Ω) is needed when the number of states increases. When Ω is adjusted in this fashion, increasing the number of states in the Tauchen (1986) method increases the precision only marginally. For instance, the relative error in approximating σ_y reduces from 0.35 percent to 0.22 percent when N increases from five to ten.

Changing the Persistence Parameter Table 6 compares the performance of the five methods under different values of ρ. The superior performance of the Rouwenhorst method is robust to changes in this parameter. In particular, increasing the persistence of the AR(1) process from 0.5 to 0.979 has very little impact on its precision. This shows that the Rouwenhorst method is a reliable technique for approximating stationary AR(1) process in general.

Similar to the results in Table 5, the parameter Ω in the Tauchen (1986) method is adjusted in each case so as to match the actual value of σ_a. The resulting values are shown in Table 6. In general, a wider state space (i.e., a larger value of Ω) is needed for less persistent processes. When Ω is calibrated to match σ_a, the Tauchen (1986) method has better performance in approximating highly persistent processes. For instance, when $\rho = 0.5$ the relative errors in approximating σ_{ka} and ρ_y are 4.66 percent and 1.84 percent, respectively. These become 1.34 percent and 0.36 percent, respectively, when $\rho = 0.979$. The precision of this method in approximating the standard deviations is not sensitive to changes in ρ.

The performance of the quadrature-based methods is very sensitive to the value of ρ. Similar to Flodén (2008a), our results show that the quadrature-based methods work best in approximating AR(1) processes with low persistence. But unlike Flodén (2008a) which only focuses on the parameters of the AR(1) process, the current study also considers the impact of these methods on the moments of the endogenous variables. When ρ equals to 0.5 or 0.6, the original Tauchen-Hussey method and its variation can generate highly accurate approximations that are comparable to those generated by the Rouwenhorst method. The relative errors for the business cycle moments are all less than one percent. Within this range of ρ, the two quadrature-based methods are more accurate than the Tauchen (1986) method, especially in approximating σ_{ka} and ρ_y. However, the accuracies of the Tauchen-Hussey method and Flodén's variation deteriorate quickly when the persistence parameter approaches one. For instance, the Tauchen-Hussey method has a relative error of 25 percent in approximating σ_y when ρ equals to 0.9 and an error of 61 percent when ρ is 0.979.

Finally, it is worth mentioning that the results of the two experiments conducted in the error analysis section are also robust to different values of the persistence parameter. These results are summarized as follow.[9] First, the figures reported in Table 6 are largely unaffected when we replace the computed policy function with the actual one. Second, when the business cycle moments are computed using Monte Carlo simulations, all five discretization methods generate very similar results.

Changing the Standard Deviation of the White Noise Process The performance of the five methods under different values of σ_ε are shown in Table 7. In terms of approximating the AR(1) process, increasing the value of σ_ε from 0.001 to 0.1 does not seem to affect the performance of these methods. In terms of approximating the standard deviations of the endogenous variables and the covariance between a and k, the accuracies of Flodén's method and the Adda-Cooper method improve when the AR(1) process is less volatile. The opposite is true for the Rouwenhorst method and the Tauchen (1986) method. The variations in the relative errors, however, are not significant. More specifically, increasing σ_ε from 0.001 to 0.1 changes the relative errors by less than two percentage points in most cases. Finally, the precision of all five methods in approximating ρ_y is not sensitive to changes in the value of σ_ε.

[9] The numerical results are not shown in the paper but are available from the authors upon request.

Relaxing the Assumption of Full Depreciation

This section evaluates the performance of the five discretization methods in solving the stochastic growth model when the full depreciation assumption is relaxed. The rate of depreciation is now taken to be 2.5 percent, which is the same as in King and Rebelo (1999). All other parameters remain the same as in the baseline case. Under this parameterization, the business cycle moments of interest do not have closed-form solutions. Thus we first compute a highly accurate approximation of these moments. To do this we use the Chebyshev parameterized expectation algorithm described in Christiano and Fisher (2000) to compute the policy function.[10] We then generate a sequence of a_t of length 50,010,000 using the actual AR(1) process. The first 10,000 observations are discarded and the rest are used to compute the business cycle moments. The solutions obtained are taken as the "true" solutions of the model.[11]

Panel (A) of Table 8 shows the results obtained under the baseline approach. The Rouwenhorst method again has the best overall performance. Most importantly, this method is capable of producing highly accurate approximations even when N is small. The Tauchen (1986) method has the second best overall performance, followed by Flodén's variation of the Tauchen-Hussey method. However, the performances of these two methods are extremely sensitive to the size of the grid for a_t and deteriorate significantly when N decreases. Finally, it is worth noting that the five methods have very different performances in approximating the covariance between k_t and a_t, especially when N is small. A method that generates a good approximation for this statistic also tends to yield accurate approximations for the covariances between y_t and other endogenous variables. It is thus important to choose a method that can match this statistic well. As the table shows, the Rouwenhorst method generates the most accurate approximation of this covariance and, as a result, the rest of the business cycle moments.[12]

[10] Specifically, we compute the continuous shock version of the model using the Chebyshev parameterized expectation approach and the Wright-William specification of the conditional expectation function. The conditional expectation function is approximated by $\sum_{i=1}^{N} \theta_i C_i(k,a)$, where $C_i(k,a), i = 1,\ldots,N$ are the elements of the set $\{T_{i1}(\phi(k))T_{i2}(\psi(a))|\sum_{j=1}^{2} i_j \leq n\}$ and $\theta_i, i = 1,\ldots,N$ are the weights. The functions T_{ij} for $j = 1,2$ are the ith Chebyshev polynomials and ϕ and ψ are linear mappings of $[k_{min}, k_{max}]$ and $[-3\sigma, 3\sigma]$ into the interval $[-1,1]$. We set $n = 12$ so that $N = 78$ and we use $M = 2,916$ quadrature nodes, 54 in each direction. Further increasing N and M results in a less than 1 percent change in all the business cycle moments computed. Following Christiano and Fisher (2000), the conditional expectation is computed using 4-point Gauss-Hermite quadrature.

[11] The same method is used in both Christiano and Fisher (2000) and Santos and Peralta-Alva (2005) to obtain the "exact" solutions.

[12] These results are not shown in the paper to conserve space but are available from the authors upon request.

Panel (B) of Table 8 reports the simulation results. These results show that the choice of discretization method matters even when the business cycle moments are computed using Monte Carlo simulations. This is, in part, because linear interpolation is used to approximate $g(k_t, a_t)$ for values of k_t and a_t that are outside the discrete state space. The size of the error due to the interpolation procedure depends on the location of the grid points and hence the choice of the discretization method. However, as N increases, the state space becomes finer and the overall error due to interpolation decreases. For the Rouwenhorst method, a five-fold increase in N only marginally affects its precision. In fact, this method is able to produce highly accurate approximations even when $N = 5$. But for the other methods, such an increase in N generates a significant improvement in their performance. Consequently, it is only with 25 states in the Markov chain that the Tauchen (1986) method, the Tauchen-Hussey method and Flodén's variation can achieve degrees of accuracy on par with the Rouwenhorst method.

Two additional observations of Table 8 are worth noting. First, in terms of solving the stochastic growth model, value-function iteration, together with a five-state Markov chain constructed using the Rouwenhorst method, produces highly accurate results that are nearly identical to the "true" solutions computed using Chebyshev PEA. This is an important finding because, under the given parameterization, the first method requires substantially less computational time and is significantly easier to implement than the latter. Second, when comparing between the two panels of Table 2, one can see that the baseline approach, when combined with the Rouwenhorst method, can generate estimated moments that are as accurate as those produced by the simulation method with five million draws. Our results thus show that simulation is not necessary to generate accurate statistics. In fact, it may result in less accuracy than the baseline approach if the sample size is too small.[13]

3.2 Income Fluctuation Problem

Consider an infinitely-lived, risk-averse consumer who receives a random labor endowment e_t in each period t. The agent can self-insure by borrowing and lending via a single risk-free asset but there is an upper bound on how much he can borrow. Formally, the consumer's problem is given

[13] For instance, the baseline approach with the Rouwenhorst method yields more accurate statistics than the simulation method when only one million draws are used.

by

$$\max_{\{c_t,a_{t+1}\}_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty}\beta^t \log\left(c_t\right)\right],$$

subject to

$$c_t + a_{t+1} = we_t + (1+r)\,a_t,$$

$$\ln e_{t+1} = \rho \ln e_t + \varepsilon_{t+1}, \qquad \text{with } \rho \in (0,1),$$

$c_t \geq 0$, and $a_{t+1} \geq -\underline{a}$. The variable c_t denotes period t consumption, a_t denotes period t assets, w is the wage, r is the return on assets, $\underline{a} \geq 0$ is the borrowing limit and $\varepsilon_{t+1} \sim$i.i.d. $N\left(0, \sigma_\varepsilon^2\right)$.

Let $\mathcal{S}$ be the state space and $\Psi\left(a,e\right)$ be the feasible choice set for next-period assets when the current state is (a,e).[14] The Bellman equation for the consumer's problem is given by

$$V\left(a,e\right) = \max_{a' \in \Psi(a,e)} \left\{\log\left(we + (1+r)\,a - a'\right) + \beta \int V\left(a',e'\right) dF\left(e'|e\right)\right\}, \tag{12}$$

where $F\left(\cdot|e\right)$ is the distribution function of e_{t+1} conditional on $e_t = e$. The solution of this problem includes a value function $V : \mathcal{S} \to \mathbb{R}$ and a policy function $g : \mathcal{S} \to \mathbb{R}$ for next-period assets.

Parameterization and Computation

The following parameter values are used in the computation. The subjective discount factor β is chosen to be 0.96. The borrowing limit $\underline{a}$ is set to zero so that no borrowing is allowed. The rate of return r is taken to be 3.75 percent and the wage rate is normalized to one. As for the labor endowment process, we consider two different specifications that are commonly used in the literature. In the first specification, we follow Aiyagari (1994) and set $\rho = 0.90$ and $\sigma_\varepsilon = 0.2$. In the second specification, we use the estimates obtained by French (2005), which are $\rho = 0.977$ and $\sigma_\varepsilon = 0.12$.[15]

The computational procedure consists of the following steps: First, we compute a highly accurate approximation of the inequality measures of interest. Again we use the Chebyshev PEA to compute

[14] Formally, the state space is defined as $\mathcal{S} = \{(a,e) : a \geq -\underline{a} \text{ and } e > 0\}$ and the feasible choice set is defined as $\Psi\left(a,e\right) = \{a' : we + (1+r)\,a \geq a' \geq -\underline{a}\}$.

[15] Storesletten *et al.* (1999) report similar estimates for ρ and σ_ε. Pijoan-Mas (2006) uses the estimates reported in French (2005) in his calibration. Similar values for ρ and σ_ε are used in other studies including Chang and Kim (2006, 2007) and Flodén (2008b).

the policy function $a' = g(a, e)$.[16] We then generate a sequence of $\ln e_t$ of length 50,010,000 using the actual AR(1) process. The first 10,000 observations are discarded and the rest are used to compute two inequality measures for consumption, total income and assets. These measures are the coefficient of variation (CV) and the Gini coefficient. Next, we use value-function iteration with linear interpolation on the value function to solve the Bellman equation in (12) on a discrete state space $\widehat{\mathcal{S}}$. Specifically, we form a 25-state Markov chain using each of the five methods and use 1,500 grid points for assets.[17] We then compute the two inequality measures using the baseline approach and Monte Carlo simulations. In the baseline approach, we use 15,000 grid points for assets to compute the stationary distribution. In the Monte Carlo simulations, we generate a sequence of $\ln e_t$ of length 5,010,000 using the actual AR(1) process, discard the first 10,000 and use the rest to compute the inequality measures.

Results

The ratios of the inequality measures obtained under the baseline approach and the simulation approach to their "true" values are shown in Panels (A) and (B) of Table 9. The table shows that for some inequality measures, the results obtained from the discrete state space method differ significantly from the "true" values. In particular, for all five discretization methods considered, the discrete state space method tends to underestimate the degree of wealth inequality under both approaches. This problem remains even when a 25-state grid for e_t is used and arises from errors in the approximation of the policy function that occur when the domain for e_t is discretized.[18]

The table also shows that the choice of discretization method is important when using the baseline approach. Moreover, under this approach, methods that generate relatively more accurate approximations for the persistence and the standard deviation of the AR(1) process also tend to yield relatively more accurate solutions. This is consistent with the findings for the stochastic growth model. Under Aiyagari's specification of the labor endowment process, and with $N = 25$,

[16] Specifically, the same method described in footnote 10 is used to approximate the conditional expectation function. The only difference is, in this case, we set $n = 23$ so that $N = 276$ and we use $M = 40,000$, with 200 nodes in each direction. Further increasing N and M results in a less than 1 percent change in all the moments computed.

[17] We use a transformation of assets so that there are more grid points around the borrowing limit $\underline{a}$. The resulting grid points are thus not evenly spaced. This procedure is commonly used in solving the income fluctuation problem. See, for instance, den Haan (2010).

[18] Note that this result is not due to the coarseness of the asset grids as doubling their size does not improve the accuracies of these statistics.

the Tauchen-Hussey method, Flodén's variation and the Rouwenhorst method have the best performance. Under French's specification, where the AR(1) process is more persistent, Flodén's variation and the Rouwenhorst method continue to have the best performance but the accuracy of the Tauchen-Hussey method deteriorates significantly. Thus Flodén's variation and the Rouwenhorst method are more robust to variations in ρ. However, the performance of Flodén's method is rather sensitive to the choice of N. In particular, the accuracy of this method decreases considerably when N is lowered from 25 to 10. Meanwhile, the accuracy of the Rouwenhorst method is only marginally affected by this change. These findings illustrate that, under this approach, only the Rouwenhorst method is robust to changes in both N and ρ.

In contrast, under the simulation approach, all five methods yield very similar results when Aiyagari's specification is used. When ρ is increased to 0.977, larger differences in the simulation results are observed. However, in this case, no single method dominates the others in all measures. These results show that a significant amount of the variation in accuracy of the different methods under the baseline approach is due to variation in the accuracy of the discrete approximation to the stationary distribution. Comparing across the two approaches, note that while some methods perform better than the Rouwenhorst method in some cases, the Rouwenhorst method is the most consistent across the two approaches.

In sum, the choice of discretization method can have a significant impact on the accuracy of model solutions. In general, among the five methods considered, the Rouwenhorst method is found to be one of the most accurate. Moreover, it is the most robust to variations in the persistence of the exogenous process, the number of states in the Markov chain, and the approach used in obtaining the statistics from the stationary distribution.

4 Conclusions

The main contributions of this paper are two-fold. First, it re-examines the Rouwenhorst method of approximating stationary AR(1) processes and shows formally that this method can match five important statistics of any stationary AR(1) process. This property makes the Rouwenhorst method more reliable than other methods in approximating highly persistent processes. Second, it compares the performances of the Rouwenhorst method and four other methods in solving the stochastic

growth model and a standard income fluctuation problem. Our quantitative results show that the accuracy of the approximation for the exogenous process can have a large impact on the computed solutions of these models. In particular, a good approximation for the persistence and the standard deviation of the AR(1) process is important for obtaining accurate approximations of statistics generated from the models. The Rouwenhorst method has one of the best performances in these regards. This is because, unlike the other methods, it can generate relatively accurate solutions when the persistence of the exogenous process is very close to one regardless of the coarseness of the state space for the Markov chain or the approach used to compute the statistics from the stationary distribution.

Table 2 Baseline Results for the Stochastic Growth Model.

(A) Approximating the AR(1) process

	Generated Values Relative to True Values				
	Tauchen	T-H	F	A-C	R
ρ	1.0097	0.9453	1.0096	0.9993	1.0000
σ_ε	0.8167	0.8905	0.5019	1.5599	1.0000
σ_a	1.0000	0.4006	0.7742	0.9471	1.0000

(B) Approximating the Variance-Covariance Matrix for State Variables

	Generated Values Relative to True Values				
	Tauchen	T-H	F	A-C	R
σ_k	1.0053	0.3882	0.7734	0.9330	0.9986
σ_{ka}	1.0134	0.1401	0.6071	0.8464	0.9986

(C) Approximating Business Cycle Moments

	Generated Values Relative to True Values				
	Tauchen	T-H	F	A-C	R
σ_y	1.0035	0.3880	0.7763	0.9338	0.9995
σ_c	1.0026	0.3879	0.7776	0.9343	1.0000
σ_i	1.0053	0.3882	0.7734	0.9330	0.9986
ρ_y	1.0036	0.9538	1.0063	0.9807	1.0000

T-H stands for the original Tauchen-Hussey method; F stands for the variation of T-H; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, σ_ε= 0.0072, $\rho = 0.979$, $N = 5$, $M = 1.6425$.

Table 3 Error Analysis for the Stochastic Growth Model

(A) Using Computed Policy Function (Baseline case)

	Generated Values Relative to True Values				
	Tauchen	T-H	F	A-C	R
ρ	1.0097	0.9453	1.0096	0.9993	1.0000
σ_ε	0.8167	0.8905	0.5019	1.5599	1.0000
σ_a	1.0000	0.4006	0.7742	0.9471	1.0000
σ_k	1.0053	0.3882	0.7734	0.9330	0.9986
σ_{ka}	1.0134	0.1401	0.6071	0.8464	0.9986
σ_y	1.0035	0.3880	0.7763	0.9338	0.9995
σ_c	1.0026	0.3879	0.7776	0.9343	1.0000
σ_i	1.0053	0.3882	0.7734	0.9330	0.9986
ρ_y	1.0036	0.9538	1.0063	0.9807	1.0000

(B) Using Actual Policy Function

	Generated Values Relative to True Values				
	Tauchen	T-H	F	A-C	R
ρ	1.0097	0.9453	1.0096	0.9993	1.0000
σ_ε	0.8167	0.8905	0.5019	1.5599	1.0000
σ_a	1.0000	0.4006	0.7742	0.9471	1.0000
σ_k	1.0026	0.3880	0.7777	0.9343	1.0000
σ_{ka}	1.0107	0.1400	0.6104	0.8475	1.0000
σ_y	1.0026	0.3879	0.7777	0.9343	1.0000
σ_c	1.0026	0.3879	0.7777	0.9343	1.0000
σ_i	1.0026	0.3880	0.7777	0.9343	1.0000
ρ_y	1.0036	0.9537	1.0063	0.9807	1.0000

T-H stands for the original Tauchen-Hussey method; F stands for the variation of T-H; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$, $\rho = 0.979$, $N = 5$, $M = 1.6425$.

Table 4 Baseline Approach vs. Monte Carlo Simulations

(A) Baseline case

	Generated Values Relative to True Values				
	Tauchen	T-H	F	A-C	R
ρ	1.0097	0.9453	1.0096	0.9993	1.0000
σ_ε	0.8167	0.8905	0.5019	1.5599	1.0000
σ_a	1.0000	0.4006	0.7742	0.9471	1.0000
σ_k	1.0053	0.3882	0.7734	0.9330	0.9986
σ_{ka}	1.0134	0.1401	0.6071	0.8464	0.9986
σ_y	1.0035	0.3880	0.7763	0.9338	0.9995
σ_c	1.0026	0.3879	0.7776	0.9343	1.0000
σ_i	1.0053	0.3882	0.7734	0.9330	0.9986
ρ_y	1.0036	0.9538	1.0063	0.9807	1.0000

(B) Monte Carlo Simulations

	Generated Values Relative to True Values				
	Tauchen	T-H	F	A-C	R
ρ	1.0000	1.0000	1.0000	1.0000	1.0000
σ_ε	1.0000	1.0000	1.0000	1.0000	1.0000
σ_a	1.0005	1.0005	1.0005	1.0005	1.0005
σ_k	1.0005	1.0006	1.0006	1.0007	1.0006
σ_{ka}	1.0011	1.0011	1.0012	1.0013	1.0011
σ_y	1.0005	1.0005	1.0006	1.0006	1.0005
σ_c	1.0005	1.0005	1.0006	1.0006	1.0005
σ_i	1.0005	1.0006	1.0006	1.0007	1.0006
ρ_y	1.0000	1.0000	1.0000	1.0000	1.0000

T-H stands for the original Tauchen-Hussey method; F stands for the variation of T-H; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$, $\rho = 0.979$, $N = 5$, $M = 1.6425$.

Table 5 Robustness Check for the Stochastic Growth Model: Changing N.

	$N = 2$					$N = 5$ (Baseline)					$N = 10$				
	Generated Values Relative to True Values					Generated Values Relative to True Values					Generated Values Relative to True Values				
	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
ρ	1.0214	0.7688	1.0206	0.8879	1.0000	1.0097	0.9453	1.0096	0.9993	1.0000	0.9989	0.9867	1.0006	1.0038	1.0000
σ_ε	0.0087	0.6584	0.0805	1.9346	1.0000	0.8167	0.8905	0.5019	1.5599	1.0000	1.1318	0.9493	0.8886	1.2781	1.0000
σ_a	1.0000	0.2039	0.4071	0.7979	1.0000	1.0000	0.4006	0.7742	0.9471	1.0000	1.0000	0.5860	0.9558	0.9793	1.0000
σ_k	1.0040	0.1844	0.4095	0.7718	0.9966	1.0053	0.3882	0.7734	0.9330	0.9986	0.9978	0.5794	0.9583	0.9753	0.9988
σ_{ka}	1.0281	0.0283	0.1705	0.5400	0.9966	1.0134	0.1401	0.6071	0.8464	0.9986	0.9910	0.3260	0.9190	0.9386	0.9988
σ_y	1.0065	0.1870	0.4099	0.7682	0.9989	1.0035	0.3880	0.7763	0.9338	0.9995	0.9978	0.5788	0.9573	0.9743	0.9996
σ_c	1.0078	0.1882	0.4101	0.7664	1.0000	1.0026	0.3879	0.7776	0.9343	1.0000	0.9978	0.5784	0.9569	0.9739	1.0000
σ_i	1.0040	0.1844	0.4095	0.7718	0.9966	1.0053	0.3882	0.7734	0.9330	0.9986	0.9978	0.5794	0.9583	0.9753	0.9988
ρ_y	1.0107	0.8752	1.0103	0.9422	1.0000	1.0036	0.9538	1.0063	0.9807	1.0000	0.9969	0.9817	1.0015	0.9922	1.0000

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F stands for the variation of T-H;

A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$, $\rho = 0.979$.

* For the Tauchen (1986) method, $M = 1.0000$ when $N = 2$, $M = 1.6425$ when $N = 5$ and $M = 1.9847$ when $N = 10$.

Table 6 Robustness Check for the Stochastic Growth Model: Changing ρ.

	$\rho = 0.5$					$\rho = 0.6$					$\rho = 0.7$				
	Generated Values Relative to True Values					Generated Values Relative to True Values					Generated Values Relative to True Values				
	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
ρ	0.9680	0.9997	1.0000	0.9310	1.0000	0.9725	0.9986	0.9999	0.9471	1.0000	0.9774	0.9953	0.9997	0.9665	1.0000
σ_ε	1.0129	0.9994	0.9999	0.9737	1.0000	1.0207	0.9972	0.9993	0.9888	1.0000	1.0331	0.9905	0.9969	1.0112	1.0000
σ_a	1.0000	0.9990	0.9999	0.9471	1.0000	1.0000	0.9950	0.9993	0.9471	1.0000	1.0000	0.9793	0.9963	0.9471	1.0000
σ_k	0.9941	1.0025	1.0048	0.9364	1.0010	0.9934	0.9973	1.0017	0.9345	0.9991	0.9934	0.9788	0.9969	0.9243	0.9983
σ_{ka}	0.9534	0.9995	1.0035	0.8050	1.0011	0.9537	0.9873	0.9998	0.8072	0.9981	0.9552	0.9456	0.9925	0.8050	0.9985
σ_y	0.9935	0.9996	1.0009	0.9336	1.0002	0.9924	0.9948	0.9997	0.9316	0.9996	0.9918	0.9769	0.9963	0.9282	0.9996
σ_c	0.9934	0.9983	0.9992	0.9324	1.0001	0.9922	0.9938	0.9989	0.9304	1.0000	0.9911	0.9761	0.9962	0.9302	1.0003
σ_i	0.9941	1.0025	1.0048	0.9364	1.0010	0.9934	0.9973	1.0017	0.9345	0.9991	0.9934	0.9788	0.9969	0.9243	0.9983
ρ_y	0.9816	1.0002	1.0009	0.9588	1.0001	0.9818	0.9985	1.0003	0.9599	0.9999	0.9825	0.9943	0.9997	0.9613	0.9998

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F stands for the variation of T-H; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$, $N = 5$.

* For the Tauchen (1986) method, $M = 1.9241$ when $\rho = 0.5$, $M = 1.9128$ when $\rho = 0.6$ and $M = 1.8917$ when $\rho = 0.7$.

Table 6 (continued) Robustness Check for the Stochastic Growth Model: Changing ρ.

	$\rho = 0.9$					$\rho = 0.95$					$\rho = 0.979$ (baseline)				
	Generated Values Relative to True Values					Generated Values Relative to True Values					Generated Values Relative to True Values				
	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
ρ	0.9884	0.9689	0.9986	1.0060	1.0000	0.9981	0.9550	1.0025	1.0067	1.0000	1.0097	0.9453	1.0096	0.9993	1.0000
σ_ε	1.1027	0.9379	0.9379	1.1403	1.0000	1.0964	0.9101	0.8142	1.2822	1.0000	0.8167	0.8905	0.5019	1.5599	1.0000
σ_a	1.0000	0.7701	0.9347	0.9471	1.0000	1.0000	0.5904	0.8639	0.9471	1.0000	1.0000	0.4006	0.7742	0.9471	1.0000
σ_k	0.9920	0.7564	0.9337	0.9258	1.0024	0.9957	0.5753	0.8660	0.9346	0.9948	1.0053	0.3882	0.7734	0.9330	0.9986
σ_{ka}	0.9642	0.5453	0.8716	0.8248	1.0024	0.9836	0.3108	0.7530	0.8398	0.9948	1.0134	0.1401	0.6071	0.8464	0.9986
σ_y	0.9918	0.7558	0.9341	0.9291	1.0008	0.9960	0.5750	0.8658	0.9328	0.9983	1.0035	0.3880	0.7763	0.9338	0.9995
σ_c	0.9917	0.7556	0.9344	0.9307	1.0000	0.9962	0.5748	0.8657	0.9319	1.0000	1.0026	0.3879	0.7776	0.9343	1.0000
σ_i	0.9920	0.7564	0.9337	0.9258	1.0024	0.9957	0.5753	0.8660	0.9346	0.9948	1.0053	0.3882	0.7734	0.9330	0.9986
ρ_y	0.9870	0.9702	0.9996	0.9718	1.0000	0.9944	0.9602	1.0030	0.9765	0.9999	1.0036	0.9538	1.0063	0.9807	1.0000

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F stands for the variation of T-H; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$, $N = 5$.

* For the Tauchen (1986) method, $M = 1.7683$ when $\rho = 0.9$, $M = 1.6963$ when $\rho = 0.95$, $M = 1.6425$ when $\rho = 0.979$.

Table 7 Robustness Check for the Stochastic Growth Model: Changing σ_ε.

	σ_ε= 0.001					σ_ε= 0.01					σ_ε= 0.1				
	Generated Values Relative to True Values					Generated Values Relative to True Values					Generated Values Relative to True Values				
	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
ρ	1.0097	0.9453	1.0096	0.9993	1.0000	1.0097	0.9453	1.0096	0.9993	1.0000	1.0097	0.9453	1.0096	0.9993	1.0000
σ_ε	0.8167	0.8905	0.5019	1.5599	1.0000	0.8167	0.8905	0.5019	1.5599	1.0000	0.8167	0.8905	0.5019	1.5599	1.0000
σ_a	1.0000	0.4006	0.7742	0.9471	1.0000	1.0000	0.4006	0.7742	0.9471	1.0000	1.0000	0.4006	0.7742	0.9471	1.0000
σ_k	1.0265	0.4166	0.7911	0.9546	0.9910	1.0041	0.3864	0.7798	0.9328	1.0022	1.0026	0.3879	0.7775	0.9342	1.0000
σ_{ka}	1.0342	0.1497	0.6203	0.8651	0.9905	1.0122	0.1395	0.6121	0.8462	1.0022	1.0107	0.1400	0.6103	0.8475	1.0000
σ_y	1.0103	0.3965	0.7819	0.9405	0.9969	1.0031	0.3875	0.7784	0.9338	1.0007	1.0026	0.3879	0.7776	0.9342	1.0000
σ_c	1.0028	0.3875	0.7778	0.9340	1.0000	1.0026	0.3880	0.7777	0.9343	1.0000	1.0026	0.3879	0.7777	0.9343	1.0000
σ_i	1.0265	0.4166	0.7911	0.9546	0.9910	1.0041	0.3864	0.7798	0.9328	1.0022	1.0026	0.3879	0.7775	0.9342	1.0000
ρ_y	1.0037	0.9554	1.0063	0.9810	0.9999	1.0036	0.9536	1.0063	0.9807	1.0000	1.0036	0.9537	1.0063	0.9807	1.0000

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F stands for the variation of T-H; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, ρ= 0.979, $N = 5$.

* For the Tauchen (1986) method, $M = 1.6425$ in all three cases.

Table 8 Business Cycle Moments for the Stochastic Growth Model when $\delta = 0.025$.

(A) Baseline Approach

	$N = 5$					$N = 10$					$N = 25$				
	Generated Values Relative to True Values					Generated Values Relative to True Values					Generated Values Relative to True Values				
	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
ρ	1.0097	0.9453	1.0096	0.9993	1.0000	0.9989	0.9867	1.0006	1.0038	1.0000	0.9997	0.9980	1.0000	1.0012	1.0000
σ_ε	0.8167	0.8905	0.5019	1.5599	1.0000	1.1318	0.9493	0.8886	1.2781	1.0000	1.0389	0.9877	0.9994	1.0958	1.0000
σ_a	1.0000	0.4006	0.7742	0.9471	1.0000	1.0000	0.5860	0.9558	0.9793	1.0000	1.0000	0.8481	0.9996	0.9937	1.0000
σ_k	1.0060	0.3332	0.7485	0.8880	0.9980	0.9966	0.5497	0.9642	0.9598	1.0100	1.0057	0.8442	1.0069	0.9936	1.0055
σ_{ka}	1.0733	0.0810	0.6528	0.6629	0.9981	0.9494	0.2557	0.9428	0.8524	1.0107	0.9922	0.6770	1.0058	0.9588	1.0031
σ_y	1.0150	0.3515	0.7847	0.8904	0.9995	0.9897	0.5516	0.9629	0.9555	1.0033	0.9992	0.8379	1.0018	0.9881	1.0013
σ_c	1.0523	0.2905	0.8423	0.7949	1.0055	0.9719	0.5008	0.9792	0.9153	1.0071	0.9961	0.8199	1.0053	0.9779	1.0052
σ_i	0.9321	0.6555	0.6549	1.2853	1.0253	1.0944	0.8007	0.9473	1.1497	1.0389	1.0521	0.9527	1.0304	1.0713	1.0277
ρ_y	1.0037	0.9412	1.0061	0.9779	1.0000	0.9968	0.9790	1.0015	0.9915	1.0001	0.9991	0.9959	1.0000	0.9975	1.0000

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F stands for the variation of T-H; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 0.025$, $\alpha = 0.33$, $\beta = 0.984$, ρ= 0.979, σ_ε= 0.0072.

* For the Tauchen (1986) method, $M = 1.6425$ when $N = 5$, $M = 1.9847$ when $N = 10$ and $M = 2.5107$ when $N = 25$.

Table 8 Business Cycle Moments for the Stochastic Growth Model when $\delta = 0.025$.

(B) Monte Carlo Simulations

	$N = 5$					$N = 10$					$N = 25$				
	Generated Values Relative to True Values					Generated Values Relative to True Values					Generated Values Relative to True Values				
	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
ρ	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
σ_ε	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
σ_a	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005	1.0005
σ_k	0.9508	0.5439	0.8194	1.1524	1.0012	1.0450	0.8066	0.9801	1.0682	1.0114	1.0122	0.9968	1.0075	1.0288	1.0085
σ_{ka}	0.9525	0.5185	0.8202	1.1516	0.9993	1.0434	0.7925	0.9803	1.0682	1.0124	1.0125	0.9962	1.0076	1.0289	1.0091
σ_y	0.9853	0.8540	0.9442	1.0482	1.0003	1.0141	0.9372	0.9941	1.0218	1.0041	1.0042	0.9992	1.0027	1.0094	1.0031
σ_c	1.0002	0.9965	0.9990	1.0253	1.0051	1.0094	1.0071	1.0026	1.0124	1.0052	1.0049	1.0089	1.0034	1.0071	1.0043
σ_i	0.9790	0.7608	0.9103	1.1777	1.0248	1.0645	0.9428	1.0033	1.0887	1.0312	1.0293	1.0401	1.0205	1.0474	1.0255
ρ_y	0.9997	0.9961	0.9987	1.0009	1.0000	1.0003	0.9985	0.9999	1.0004	1.0001	1.0001	1.0000	1.0001	1.0002	1.0001

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F stands for the variation of T-H; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 0.025$, $\alpha = 0.33$, $\beta = 0.984$, ρ= 0.979, σ_ε= 0.0072.

* For the Tauchen (1986) method, $M = 1.6425$ when $N = 5$, $M = 1.9847$ when $N = 10$ and $M = 2.5107$ when $N = 25$.

Table 9 Measures of Inequality for the Income Fluctuation Problem

(A) Baseline Approach

		$N = 10$					$N = 25$				
		Generated Values Relative to True Values					Generated Values Relative to True Values				
Aiyagari (1994) values: $\rho = 0.9$, $\sigma = 0.2$											
		Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
AR(1) process	ρ	0.9978	0.9976	0.9999	1.0087	1.0000	0.9996	1.0000	1.0000	1.0024	1.0000
	σ	1.0000	0.9462	0.9969	0.9793	1.0000	1.0000	0.9996	1.0000	0.9937	1.0000
Labor Endowment (e_t)	CV	0.9725	0.9187	0.9921	0.9402	0.9862	0.9890	0.9996	1.0010	0.9734	0.9954
Consumption (c_t)	CV	0.9038	0.8631	0.8555	0.8500	0.9497	0.9435	0.9641	0.9659	0.9131	0.9598
	Gini	0.9112	0.8709	0.9511	0.8621	0.9479	0.9433	0.9569	0.9579	0.9194	0.9541
Total Income ($we_t + ra_t$)	CV	0.9162	0.8783	0.9496	0.8801	0.9430	0.9418	0.9571	0.9586	0.9211	0.9527
	Gini	0.9554	0.9176	0.9707	0.9284	0.9701	0.9685	0.9746	0.9752	0.9560	0.9733
Assets (a_t)	CV	0.6487	0.6571	0.6877	0.6271	0.6793	0.6759	0.6928	0.6939	0.6574	0.6884
	Gini	0.7626	0.7707	0.7906	0.7453	0.7849	0.7829	0.7931	0.7934	0.7707	0.7902
French (2005) values: $\rho = 0.977$, $\sigma = 0.12$											
		Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
AR(1) process	ρ	0.9987	0.9872	1.0004	1.0040	1.0000	0.9997	0.9982	1.0000	1.0013	1.000
	σ	1.0000	0.6084	0.9587	0.9793	1.0000	1.000	0.8683	0.9996	0.9937	1.000
Labor Endowment (e_t)	CV	0.9365	0.5594	0.9111	0.9203	0.9794	0.9665	0.8131	0.9999	0.9622	0.9937
Consumption (c_t)	CV	0.8352	0.4906	0.8878	0.7884	0.9331	0.9021	0.7581	0.9529	0.8807	0.9465
	Gini	0.9079	0.5471	0.9392	0.8631	0.9694	0.9548	0.8264	0.9771	0.9410	0.9746
Total Income ($we_t + ra_t$)	CV	0.8540	0.5363	0.8813	0.8229	0.9351	0.9101	0.7759	0.9556	0.8962	0.9490
	Gini	0.9514	0.6185	0.9578	0.9216	0.9921	0.9831	0.8669	0.9988	0.9741	0.9967
Assets (a_t)	CV	0.5026	0.5002	0.5851	0.4814	0.5885	0.5613	0.5337	0.6021	0.5486	0.5974
	Gini	0.7229	0.7246	0.7986	0.7005	0.7904	0.7726	0.7566	0.7985	0.7651	0.7956

Notation: ρ and σ are the persistence and the standard deviation of $\ln e_t$. Parameter values: $\beta = 0.96$, $r = 0.0375$, $w = 1$.

* Under the Aiyagari (1994) calibration, Ω= 2.2540 when $N = 10$ and Ω= 2.8176 when $N = 25$.Under the French (2005) calibration, Ω= 1.9986 when $N = 10$ and Ω= 2.5307 when $N = 25$.

Table 9 Measures of Inequality for the Income Fluctuation Problem

(B) Monte Carlo Simulations

		$N = 10$					$N = 25$				
		Generated Values Relative to True Values					Generated Values Relative to True Values				
Aiyagari (1994) values: $\rho = 0.9$, $\sigma = 0.2$											
		Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
AR(1) process	ρ	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	σ	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
Labor Endowment (e_t)	CV	0.9985	0.9985	0.9985	0.9985	0.9985	0.9985	0.9985	0.9985	0.9985	0.9985
Consumption (c_t)	CV	0.9636	0.9682	0.9612	0.9625	0.9602	0.9634	0.9626	0.9623	0.9619	0.9621
	Gini	0.9559	0.9605	0.9537	0.9551	0.9528	0.9554	0.9546	0.9543	0.9544	0.9541
Total Income ($we_t + ra_t$)	CV	0.9554	0.9579	0.9526	0.9558	0.9513	0.9562	0.9554	0.9548	0.9556	0.9545
	Gini	0.9740	0.9771	0.9711	0.9751	0.9701	0.9738	0.9730	0.9725	0.9737	0.9723
Assets (a_t)	CV	0.6856	0.6925	0.6791	0.6906	0.6751	0.6917	0.6901	0.6875	0.6913	0.6864
	Gini	0.7870	0.7954	0.7806	0.7941	0.7770	0.7912	0.7897	0.7874	0.7925	0.7864
French (2005) values: $\rho = 0.977$, $\sigma = 0.12$											
		Tau*	T-H	F	A-C	R	Tau*	T-H	F	A-C	R
AR(1) process	ρ	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	σ	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988
Labor Endowment (e_t)	CV	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988
Consumption (c_t)	CV	0.9682	1.0355	0.9204	0.9812	0.9479	0.9632	0.9928	0.9520	0.9721	0.9511
	Gini	0.9926	1.0683	0.9489	1.0076	0.9727	0.9838	1.0156	0.9751	0.9917	0.9745
Total Income ($we_t + ra_t$)	CV	0.9675	1.0198	0.9231	0.9780	0.9501	0.9645	0.9887	0.9546	0.9722	0.9536
	Gini	1.0139	1.0809	0.9727	1.0274	0.9949	1.0051	1.0342	0.9971	1.0119	0.9965
Assets (a_t)	CV	0.5864	0.5981	0.5790	0.5862	0.5855	0.6050	0.6122	0.6007	0.6115	0.5969
	Gini	0.7860	0.8117	0.7851	0.7899	0.7836	0.7984	0.8156	0.7965	0.8048	0.7932

Notation: ρ and σ are the persistence and the standard deviation of $\ln e_t$. Parameter values: $\beta = 0.96$, $r = 0.0375$, $w = 1$.

* Under the Aiyagari (1994) calibration, Ω= 2.2540 when $N = 10$ and Ω= 2.8176 when $N = 25$.Under the French (2005) calibration, Ω= 1.9986 when $N = 10$ and Ω= 2.5307 when $N = 25$.

Appendix A

The objective of this section is to derive a set of equations that can be used to describe the elements in Π_N. The proof of Proposition 1 is built upon these equations.

To begin with, the elements in the first and the last rows of Π_N can be obtained by expanding the polynomials $[p+(1-p)t]^{N-1}$ and $(1-q+qt)^{N-1}$, respectively. Using the binomial formula, we can obtain

$$\pi_{1,j}^{(N)} = \binom{N-1}{j-1} p^{N-j} (1-p)^{j-1}, \tag{13}$$

and

$$\pi_{N,j}^{(N)} = \binom{N-1}{j-1} (1-q)^{N-j} q^{j-1}, \tag{14}$$

for $j = 1, 2, ..., N$.

For all other rows, i.e., $i = 2, ..., N-1$, the elements in Π_N can be defined recursively using the elements in Π_{N-1}. Begin with the system for $N-1 \geq 2$. The system of polynomials is given by

$$\Phi(t; N-1, i) = [p+(1-p)t]^{N-1-i} (1-q+qt)^{i-1} = \sum_{j=1}^{N-1} \pi_{i,j}^{(N-1)} t^{j-1},$$

for $i = 1, ..., N-1$. There are two ways to relate this system to the one for N:

$$\Phi(t; N, i) = [p+(1-p)t] \Phi(t; N-1, i), \tag{15}$$

for $i = 1, ..., N-1$, and

$$\Phi(t; N, i) = (1-q+qt) \Phi(t; N-1, i-1), \tag{16}$$

for $i = 2, ..., N$. Substituting (3) into (15) gives

$$\begin{aligned} \sum_{j=1}^{N} \pi_{i,j}^{(N)} t^{j-1} &= [p+(1-p)t] \sum_{j=1}^{N-1} \pi_{i,j}^{(N-1)} t^{j-1} \\ &= \sum_{j=1}^{N-1} p \pi_{i,j}^{(N-1)} t^{j-1} + \sum_{j=1}^{N-1} (1-p) \pi_{i,j}^{(N-1)} t^{j}, \end{aligned}$$

for $i = 1, ..., N-1$. Similarly, substituting (3) into (16) would give

$$\begin{aligned}\sum_{j=1}^{N} \pi_{i,j}^{(N)} t^{j-1} &= (1-q+qt) \sum_{j=1}^{N-1} \pi_{(i-1),j}^{(N-1)} t^{j-1} \\ &= \sum_{j=1}^{N-1} (1-q)\, \pi_{(i-1),j}^{(N-1)} t^{j-1} + \sum_{j=1}^{N-1} q\pi_{(i-1),j}^{(N-1)} t^{j},\end{aligned}$$

for $i = 2, ..., N$. The following can be obtained by comparing the coefficients for $i = 1, 2, ..., N-1$,

$$\pi_{i,1}^{(N)} = p\pi_{i,1}^{(N-1)} = (1-q)\, \pi_{(i-1),1}^{(N-1)} \tag{17}$$

$$\begin{aligned}\pi_{i,j}^{(N)} &= p\pi_{i,j}^{(N-1)} + (1-p)\, \pi_{i,(j-1)}^{(N-1)} \\ &= (1-q)\, \pi_{(i-1),j}^{(N)} + q\pi_{(i-1),(j-1)}^{(N)}, \quad \text{for } j = 2, ..., N-1,\end{aligned} \tag{18}$$

and

$$\pi_{i,N}^{(N)} = (1-p)\, \pi_{i,(N-1)}^{(N-1)} = q\pi_{(i-1),N}^{(N-1)}. \tag{19}$$

Appendix B

Proof of Proposition 1

Fix $N \geq 2$. The elements in the Rouwenhorst matrix $\Theta_N = \left[\theta_{i,j}^{(N)}\right]$ are governed by the following sets of equations:

For the elements in the first row,

$$\theta_{1,j}^{(N)} = \begin{cases} p\theta_{1,j}^{(N-1)} & \text{if } j = 1 \\ p\theta_{1,j}^{(N-1)} + (1-p)\, \theta_{1,(j-1)}^{(N-1)} & \text{if } j = 2, ..., N-1 \\ (1-p)\, \theta_{1,(j-1)}^{(N-1)} & \text{if } j = N. \end{cases} \tag{20}$$

For the elements in the final row,

$$\theta_{N,j}^{(N)} = \begin{cases} (1-q)\,\theta_{(N-1),j}^{(N-1)} & \text{if } j = 1 \\ (1-q)\,\theta_{(N-1),j}^{(N-1)} + q\theta_{(N-1),(j-1)}^{(N-1)} & \text{if } j = 2, ..., N-1 \\ q\theta_{(N-1),(j-1)}^{(N-1)} & \text{if } j = N. \end{cases} \tag{21}$$

For the elements in row $i = 2, ..., N-1$,

$$\theta_{i,j}^{(N)} = \begin{cases} \frac{1}{2}\left[p\theta_{i,j}^{(N-1)} + (1-q)\,\theta_{(i-1),j}^{(N-1)}\right] & \text{if } j = 1 \\ \frac{1}{2}\left[(1-p)\,\theta_{i,(j-1)}^{(N-1)} + q\theta_{(i-1),(j-1)}^{(N-1)}\right] & \text{if } j = N, \end{cases} \tag{22}$$

and for $j = 2, ..., N-1$,

$$\begin{aligned} \theta_{i,j}^{(N)} &= \frac{1}{2}\left[p\theta_{i,j}^{(N-1)} + (1-p)\,\theta_{i,(j-1)}^{(N-1)} + (1-q)\,\theta_{(i-1),j}^{(N-1)}\right. \\ &\quad \left. + q\theta_{(i-1),(j-1)}^{(N-1)}\right], \end{aligned} \tag{23}$$

For any given Θ_{N-1}, the system of equations (20)-(23) defines a unique Θ_N. Similarly, for any given Π_{N-1}, the system of equations (13)-(19) defines a unique Π_N. Since $\Theta_2 = \Pi_2$, it suffice to show that the elements in Π_N generated by (13)-(19) satisfies the system (20)-(23).

Consider the first row (i.e., $i = 1$) in Π_N. According to (13),

$$\pi_{11}^{(N)} = p^{N-1} = p\pi_{11}^{(N-1)},$$

and

$$\pi_{1,N}^{(N)} = (1-p)^{N-1} = (1-p)\,\pi_{1,(N-1)}^{(N-1)}.$$

For $j = 2, ..., N-1$, since

$$\pi_{1,j}^{(N-1)} = \binom{N-2}{j-1} p^{N-1-j}\,(1-p)^{j-1},$$

$$\pi_{1,(j-1)}^{(N-1)} = \binom{N-2}{j-2} p^{N-j}\,(1-p)^{j-2},$$

and

$$\binom{N-1}{j-1} = \binom{N-2}{j-1} + \binom{N-2}{j-2},$$

we have

$$\pi_{1,j}^{(N)} = p\pi_{1,j}^{(N-1)} + (1-p)\,\pi_{1,(j-1)}^{(N-1)}.$$

This shows that the elements in the first row of Π_N satisfies (20). Using (14) and the same procedure, one can show that the elements in the last row of Π_N satisfies (21).

The rest of the proof follows immediately from (17)-(19). For any row $i = 2, ..., N-1$ in Π_N, (17) implies

$$\pi_{i,1}^{(N)} = \frac{1}{2}\left[p\pi_{i,1}^{(N-1)} + (1-q)\,\pi_{(i-1),1}^{(N-1)}\right].$$

Similarly, (18) and (19) imply

$$\pi_{i,N}^{(N)} = \frac{1}{2}\left[(1-p)\,\pi_{i,(N-1)}^{(N-1)} + q\pi_{(i-1),N}^{(N-1)}\right],$$

and

$$\begin{aligned}\pi_{ij}^{(N)} &= \frac{1}{2}\left[p\pi_{ij}^{(N-1)} + (1-p)\,\pi_{i,(j-1)}^{(N-1)} + (1-q)\,\pi_{(i-1),j}^{(N-1)}\right. \\ &\quad \left. + q\pi_{(i-1),(j-1)}^{(N-1)}\right], \end{aligned} \tag{24}$$

for $j = 2, ..., N-1$, respectively. Thus all the elements in row $i = 2, ..., N-1$ in Π_N satisfies (22) and (23). This completes the proof.

Proof of Lemma 2

It suffice to check that all the elements of Π_N are strictly positive. From (13) and (14), it is obvious that the elements in the first and the last rows are strictly positive. For the other rows, a simple induction argument is used. First, Π_2 is a stochastic matrix with non-zero entries. Suppose the result is true for $N-1 \geq 2$. It follows from (17)-(19) that $\pi_{ij}^{(N)} > 0$ for $i = 2, ..., N-1$ and for $j = 1, 2, ..., N$. This completes the proof.

Proof of Proposition 4

As mentioned in the proof of Proposition 1, the first column of Π_N is given by

$$\pi_{i,1}^{(N)} = p^{N-i}\left(1-q\right)^{i-1},$$

for $i = 1, 2, ..., N$. Define $\widehat{\lambda}_i^{(N)}$ as in (4). Then

$$\begin{aligned}
\sum_{i=1}^{N} \widehat{\lambda}_i^{(N)} \pi_{i,1}^{(N)} &= \sum_{i=1}^{N} \binom{N-1}{i-1} s^{N-i}\left(1-s\right)^{i-1} p^{N-i}\left(1-q\right)^{i-1} \\
&= \sum_{i=1}^{N} \binom{N-1}{i-1} \left(sp\right)^{N-i}\left(1-s\right)^{i-1}\left(1-q\right)^{i-1} \\
&= \left[sp + \left(1-s\right)\left(1-q\right)\right]^{N} \\
&= s^N = \widehat{\lambda}_1^{(N)}.
\end{aligned}$$

For all other columns except the first one, an induction argument is used to prove the result. As mentioned in the text, the guess is correct when $N = 2$. Suppose the guess is correct for some $N \geq 2$, i.e.,

$$\widehat{\lambda}_j^{(N)} = \sum_{i=1}^{N} \widehat{\lambda}_i^{(N)} \pi_{i,j}^{(N)}, \qquad \text{for } j = 1, 2, ..., N. \tag{25}$$

We have already proved that this is true when $j = 1$, so proceeds to $j = 2, ..., N+1$.

Using (4), the following can be derived

$$\widehat{\lambda}_i^{(N+1)} = \begin{cases} s\widehat{\lambda}_i^{(N)} & \text{for } i = 1 \\ s\widehat{\lambda}_i^{(N)} + \left(1-s\right)\widehat{\lambda}_{i-1}^{(N)} & \text{for } i = 2, ..., N, \\ \left(1-s\right)\widehat{\lambda}_{i-1}^{(N)} & \text{for } i = N+1. \end{cases} \tag{26}$$

Using these one can obtain

$$\begin{aligned}
&\sum_{i=1}^{N+1}\widehat{\lambda}_i^{(N+1)}\pi_{i,j}^{(N+1)}\\
=\;&\widehat{\lambda}_1^{(N+1)}\pi_{1,j}^{(N+1)}+\sum_{i=2}^{N}\widehat{\lambda}_i^{(N+1)}\pi_{i,j}^{(N+1)}+\widehat{\lambda}_{N+1}^{(N+1)}\pi_{(N+1),j}^{(N+1)}\\
=\;&s\widehat{\lambda}_1^{(N)}\pi_{1,j}^{(N+1)}+\sum_{i=2}^{N}\left[s\widehat{\lambda}_i^{(N)}+(1-s)\,\widehat{\lambda}_{i-1}^{(N)}\right]\pi_{i,j}^{(N+1)}+(1-s)\,\widehat{\lambda}_N^{(N+1)}\pi_{(N+1),j}^{(N+1)}\\
=\;&\sum_{i=1}^{N}s\widehat{\lambda}_i^{(N)}\pi_{i,j}^{(N+1)}+\sum_{i=1}^{N-1}(1-s)\,\widehat{\lambda}_i^{(N)}\pi_{(i+1),j}^{(N+1)}+(1-s)\,\widehat{\lambda}_N^{(N+1)}\pi_{(N+1),j}^{(N+1)}\\
=\;&\sum_{i=1}^{N}s\widehat{\lambda}_i^{(N)}\pi_{i,j}^{(N+1)}+\sum_{i=1}^{N}(1-s)\,\widehat{\lambda}_i^{(N)}\pi_{(i+1),j}^{(N+1)}. \qquad (27)
\end{aligned}$$

Based on (18), the following can be obtained

$$\pi_{i,j}^{(N+1)}=p\pi_{i,j}^{(N)}+(1-p)\,\pi_{i,j-1}^{(N)},$$

and

$$\pi_{i+1,j}^{(N+1)}=(1-q)\,\pi_{i,j}^{(N)}+q\pi_{i,(j-1)}^{(N)},$$

for $j=2,3,...,N$. Substituting these into (27) gives

$$\begin{aligned}
&\sum_{i=1}^{N+1}\widehat{\lambda}_i^{(N+1)}\pi_{i,j}^{(N+1)}\\
=\;&s\sum_{i=1}^{N}\widehat{\lambda}_i^{(N)}\left[p\pi_{i,j}^{(N)}+(1-p)\,\pi_{i,(j-1)}^{(N)}\right]+(1-s)\sum_{i=1}^{N}\widehat{\lambda}_i^{(N)}\left[(1-q)\,\pi_{i,j}^{(N)}+q\pi_{i,(j-1)}^{(N)}\right]\\
=\;&[sp+(1-s)\,(1-q)]\sum_{i=1}^{N}\widehat{\lambda}_i^{(N)}\pi_{i,j}^{(N)}+[s\,(1-p)+(1-s)\,q]\sum_{i=1}^{N}\widehat{\lambda}_i^{(N)}\pi_{i,(j-1)}^{(N)}.
\end{aligned}$$

Using the induction hypothesis (25), the following can be obtained

$$\begin{aligned}
\sum_{i=1}^{N+1}\widehat{\lambda}_i^{(N+1)}\pi_{i,j}^{(N+1)}&=[sp+(1-s)\,(1-q)]\,\widehat{\lambda}_j^{(N)}+[s\,(1-p)+(1-s)\,q]\,\widehat{\lambda}_{j-1}^{(N)}\\
&=s\widehat{\lambda}_j^{(N)}+(1-s)\,\widehat{\lambda}_{j-1}^{(N)}\\
&=\widehat{\lambda}_j^{(N+1)},
\end{aligned}$$

for $j = 2, 3, ..., N$. The last line is obtained by using (26). Since $\sum_{i=1}^{N+1} \widehat{\lambda}_i^{(N+1)} = 1$ and $\sum_{j=1}^{N+1} \pi_{i,j}^{(N+1)} = 1$, the remaining equation

$$\sum_{i=1}^{N+1} \widehat{\lambda}_i^{(N+1)} \pi_{i,j}^{(N+1)} = \widehat{\lambda}_j^{(N+1)}, \qquad \text{for } j = N+1,$$

must be satisfied. This completes the proof.

Appendix C

The objective of this section is to derive the moments listed on Table 1. Since it is understood that these are moments for an N-state Markov chain, the notations $\pi_{i,j}^{(N)}$ and $\lambda_j^{(N)}$ are simplified to become $\pi_{i,j}$ and λ_j, respectively.

Preliminaries

The following result is used in deriving the conditional mean for the Markov chain.

Lemma 5 *For any $N \geq 2$, and for $i = 1, ..., N$,*

$$\sum_{j=1}^{N} \pi_{i,j} (j-1) = (1-p)(N-i) + (i-1) q, \tag{28}$$

$$\sum_{j=1}^{N} \pi_{1,j} (j-1)^2 = \left[\sum_{j=1}^{N} \pi_{i,j} (j-1) \right]^2 + (N-i)(1-p) p + (i-1) q (1-q). \tag{29}$$

Proof. Recall the following expression

$$[p + (1-p) t]^{N-i} (1 - q + qt)^{i-1} = \sum_{j=1}^{N} \pi_{i,j} t^{j-1}, \tag{30}$$

for $i = 1, ..., N$. Equation (28) can be obtained in two steps: (i) Differentiate both sides of (30) with respect to t. (ii) Set $t = 1$.

Equation (29) can be obtained as follows: Fix $i = 1, ..., N$. Differentiate both sides of (30) with respect to t twice and set $t = 1$. This gives

$$\begin{aligned}\sum_{j=1}^{N} \pi_{i,j} (j-1)(j-2) &= \sum_{j=1}^{N} \pi_{i,j} (j-1)^2 - \sum_{j=1}^{N} \pi_{i,j} (j-1) \\ &= [(N-i)(1-p) + (i-1)q]^2 - (N-i)(1-p)^2 - (i-1)q^2 \\ &= \left[\sum_{j=1}^{N} \pi_{i,j} (j-1)\right]^2 - (N-i)(1-p)^2 - (i-1)q^2.\end{aligned}$$

Equation (29) can be obtained by combining this and equation (28). This completes the proof of Lemma 5. ■

The following equations are useful in deriving the other moments. For a binomial distribution with parameters $N-1$ and $1-s$, the first two moments are given by

$$\sum_{i=1}^{N} \binom{N-1}{i-1} s^{N-i} (1-s)^{i-1} (i-1) = (N-1)(1-s), \tag{31}$$

$$\begin{aligned}&\sum_{i=1}^{N} \binom{N-1}{i-1} s^{N-i} (1-s)^{i-1} (i-1)^2 \\ = \;& (N-1)(1-s)s + (N-1)^2 (1-s)^2. \end{aligned} \tag{32}$$

Conditional Mean

We are now ready to compute the conditional means. Conditional on $y_t = \overline{y}_i$, the mean value of y_{t+1} is given by

$$\begin{aligned}\mathrm{E}(y_{t+1} | y_t = \overline{y}_i) &= \sum_{j=1}^{N} \pi_{i,j} \overline{y}_j = \sum_{j=1}^{N} \pi_{i,j} \left[-\psi + \frac{2\psi}{N-1}(j-1)\right] \\ &= -\psi + \frac{2\psi}{N-1} \sum_{j=1}^{N} \pi_{i,j} (j-1).\end{aligned}$$

It follows from (28) that

$$\begin{aligned}\sum_{j=1}^{N}\pi_{i,j}\left(j-1\right) &= \left(1-p\right)\left(N-i\right)+\left(i-1\right)q \\ &= \left(1-p\right)\left(N-1\right)+\left(q+p-1\right)\left(i-1\right).\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{E}\left(y_{t+1}|y_t=\overline{y}_i\right) &= -\psi+\frac{2\psi}{N-1}\left[\left(1-p\right)\left(N-1\right)+\left(q+p-1\right)\left(i-1\right)\right] \\ &= -\psi+2\psi\left(1-p\right)+\left(q+p-1\right)\frac{2\psi}{N-1}\left(i-1\right) \\ &= \left(q-p\right)\psi+\left(q+p-1\right)\overline{y}_i. \qquad (33)\end{aligned}$$

Conditional Variance

Conditional on $y_t=\overline{y}_i$, the variance of y_{t+1} is given by

$$\mathrm{var}\left(y_{t+1}|\overline{y}_i\right)=\sum_{j=1}^{N}\pi_{i,j}\overline{y}_j^2-\left(\sum_{j=1}^{N}\pi_{i,j}\overline{y}_j\right)^2,$$

where

$$\sum_{j=1}^{N}\pi_{i,j}\overline{y}_j^2=\psi^2-\frac{4\psi^2}{N-1}\sum_{j=1}^{N}\pi_{i,j}\left(j-1\right)+\frac{4\psi^2}{\left(N-1\right)^2}\sum_{j=1}^{N}\pi_{i,j}\left(j-1\right)^2,$$

and

$$\left(\sum_{j=1}^{N}\pi_{ij}\overline{y}_j\right)^2=\psi^2-\frac{4\psi^2}{N-1}\sum_{j=1}^{N}\pi_{i,j}\left(j-1\right)+\frac{4\psi^2}{\left(N-1\right)^2}\left[\sum_{j=1}^{N}\pi_{i,j}\left(j-1\right)\right]^2.$$

It follows from (29) that

$$\mathrm{var}\left(y_{t+1}|\overline{y}_i\right)=\frac{4\psi^2}{\left(N-1\right)^2}\left[\left(N-i\right)\left(1-p\right)p+\left(i-1\right)q\left(1-q\right)\right].$$

Unconditional Mean

The unconditional mean of the Markov chain is given by

$$\begin{aligned}\sum_{i=1}^{N}\lambda_i\overline{y}_i &= \sum_{i=1}^{N}\lambda_i \mathrm{E}\left(y_{t+1}|y_t=\overline{y}_i\right)\\ &= \sum_{i=1}^{N}\lambda_i\left[(q-p)\,\psi+(q+p-1)\,\overline{y}_i\right]\\ &= (q-p)\,\psi+(q+p-1)\sum_{i=1}^{N}\lambda_i\overline{y}_i.\end{aligned}$$

Hence

$$\sum_{i=1}^{N}\lambda_i\overline{y}_i=\frac{(q-p)\,\psi}{2-(p+q)}\equiv\mu. \tag{34}$$

Unconditional Second Moment

$$\begin{aligned}\sum_{i=1}^{N}\lambda_i\overline{y}_i^2 &= \sum_{i=1}^{N}\lambda_i\left[-\psi+\frac{2\psi}{N-1}(i-1)\right]^2\\ &= \sum_{i=1}^{N}\lambda_i\left[\psi^2-\frac{4\psi^2}{N-1}(i-1)+\frac{4\psi^2}{(N-1)^2}(i-1)^2\right]\\ &= \psi^2-\frac{4\psi^2}{N-1}\sum_{i=1}^{N}\lambda_i(i-1)+\frac{4\psi^2}{(N-1)^2}\sum_{i=1}^{N}\lambda_i(i-1)^2.\end{aligned}$$

Using (31) and (32), we have

$$\begin{aligned}\sum_{i=1}^{N}\lambda_i\overline{y}_i^2 &= \psi^2-4\psi^2(1-s)+\frac{4\psi^2(1-s)\,s}{N-1}+4\psi^2(1-s)^2\\ &= \psi^2\left[1-4(1-s)\,s+\frac{4(1-s)\,s}{N-1}\right].\end{aligned}$$

First-order Autocovariance

First consider the following expression,

$$\begin{aligned} \mathrm{E}\left(y_t y_{t+1}\right) &= \sum_{i=1}^{N} \lambda_i \mathrm{E}\left(y_{t+1} y_t | y_t = \overline{y}_i\right) \\ &= \sum_{i=1}^{N} \lambda_i \overline{y}_i \mathrm{E}\left(y_{t+1} | y_t = \overline{y}_i\right). \end{aligned}$$

Using (33), we have

$$\begin{aligned} \mathrm{E}\left(y_t y_{t+1}\right) &= \sum_{i=1}^{N} \lambda_i \overline{y}_i \left[(q-p)\,\psi + (q+p-1)\,\overline{y}_i\right] \\ &= (q-p)\,\psi \sum_{i=1}^{N} \lambda_i \overline{y}_i + (q+p-1) \sum_{i=1}^{N} \lambda_i \overline{y}_i^2. \end{aligned} \tag{35}$$

Let σ_y^2 be the unconditional variance of the Markov chain so that

$$\sigma_y^2 = \sum_{i=1}^{N} \lambda_i \overline{y}_i^2 - \mu^2,$$

where μ is the unconditional mean defined in (34). Substituting this into (35) gives

$$\begin{aligned} & \mathrm{E}\left(y_t y_{t+1}\right) \\ = \;& (q-p)\,\psi\mu + (q+p-1)\left[\sigma_y^2 + \mu^2\right] \\ = \;& \left[(q-p)\,\psi + (q+p-1)\,\mu\right]\mu + (q+p-1)\,\sigma_y^2, \end{aligned}$$

where

$$(q-p)\,\psi + (q+p-1)\,\mu = \frac{(q-p)\,\psi}{2-(p+q)} = \mu.$$

Hence

$$\mathrm{E}\left(y_t y_{t+1}\right) = \mu^2 + (q+p-1)\,\sigma_y^2.$$

Thus the first-order autocovariance is given by

$$\mathrm{E}\left[(y_t - \mu)\,(y_{t+1} - \mu)\right] = \mathrm{E}\left(y_t y_{t+1}\right) - \mu^2 = (q+p-1)\,\sigma_y^2.$$

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